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Multivalued Usco Functions and Stegall Spaces

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Abstract
In this article we consider the study of the G-differentiability and F-ifferentiability for convex functions, not only in the general context of topological vector spaces $(t.v.s)$, but also in the context of Banach spaces. We study a special class of Banach spaces named Stegall spaces, denoted by $\mathfrak{S}$, which is located between the Asplund F-spaces and Asplund G-spaces ($\mathcal{G}$-Asplund). We present a self-contained proof of the Stegall theorem, without appealing to the huge number of references required in some proofs available in the classical literature [(1)]. This requires a thorough study of a very special type of multivalued functions between Banach spaces known as *usco* multi-functions.

Keywords: Bornology, usco mapping, subdifferential, Asplund spaces, Stegall spaces.

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1 Introducction

In this article, we take as reference the classic book *Gâteaux Differentiability of Convex Functions and Topology: Weak Asplund Spaces* [(1)], in which Marián J. Fabian talks about the Asplund spaces. First, we define the multivalued functions as follows: Let X and Y be sets. A multivalued function from X to Y is a relation which assigns to each $x \in X$ a subset of Y, denoted $f(x)$; $f(x) = \emptyset$ for some $x \in X$ is admitted. The graph of the multivalued functions f is the set $Graph(f) := \{(x, y) \in X \times Y : y \in f(x)\}$. If for each $x \in X$, $f(x)$ has only one element, we say the function is single-valued. A single-valued function from X to Y, is a relation $x \mapsto y = f(x)$ which assigns to each $x \in X$ a unique $y \in Y$. The graph of such functions is $Graph(f) := \{(x, y) \in X \times Y : y = f(x)\}$.

The objective of this article is the study of the properties of the usco functions. A usco function is a multivalued function $f: X \to Y$ between Banach spaces which is upper semicontinuous and $f(x)$ is compact for each $x \in X$. A usco function $f: X \to Y$ is minimal if for each usco function $g: X \to Y$ such that

$$g(x) \subseteq f(x) \text{ we have that } f = g.$$

One of the most important examples of the usco functions appears in the study of differentiability of the convex functions defined on an open convex subset Ω of a Banach space X. It is convenient to note that the *subdifferential* of a function $f: \Omega \to \mathbb{R} \cup \{+\infty\}$ in $x \in \Omega$ is the multivalued function $\partial f: X \to X'$ defined as follows:

$$x' \in \partial f(x) \text{ if and only if } f(y) - f(x) \geq \langle y - x, x' \rangle \text{ for every } y \in \Omega.$$

It is also important to observe that in the completely regular topological spaces, singletons and closed sets can be separated by open sets. If X is a Banach space, the topological dual with the weak topology is a completely regular space. Let us remember that a subset G of a topological space X is *residual* if there exists a countable family $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open dense subsets such that $G \supseteq \bigcap_n \mathcal{U}_n$.

The Asplud G-spaces are Banach spaces in which every continuous convex function, defined on an open convex set Ω, is differentiable in the Gâteaux sense in a residual subset of Ω. We denote by $\mathfrak{U}_G$ the set of Asplund G-spaces. The Asplund F-spaces are Banach spaces in which every continuous convex function, is differentiable in the Fréchet sense in a residual subset of Ω. We denote by $\mathfrak{U}_F$ the set of Asplund F-spaces. A Banach space X is a Stegall space if for every Baire space Z, every usco minimal function $f: Z \to X'$ (with the weak topology) is single-valued in a residual subset. We denote by $\mathfrak{S}$ the set of Stegall spaces.

2 $\boldsymbol{\beta}$-Differentiability

Let us start with the following definition:

Definition 2.1 *Let X be a vector space on a field $\mathbb{K}$. A vector bornology in X is a subset β of parts of X, denoted by $\wp(X)$, satisfying the following axioms:*

$borv$ 1. The union of all sets of β is X:

$$\bigcup_{B\in\beta} B = X.$$

$borv$ 2. β is stable under inclusions. That is, if $B \in \beta$ and $D \subset B$, then $D \in \beta$.
$borv$ 3. β is stable under finite union. That is, if $\{S_k: 1 \leq k \leq n\} \subset \beta$, then

$$\bigcup_{k=1}^{n} S_k \in \beta.$$

$borv$ 4. β is stable under the sum operation. That is, if $\{S_k: 1 \leq k \leq n\} \subset \beta$ then

$$\sum_{k=1}^{n} S_k \in \beta.$$

$borv$ 5. β is stable under the scalar multiplication operation. That is, if $S \in \beta$ and $\lambda \in \mathbb{K}$, then $\lambda S \in \beta$.
$borv$ 6. β is stable under the formation of balanced envelope. That is, if $S \in \beta$, then $bal(S) \in \beta$. Where $bal(S)$ denote the balanced hull of a set S defined by

$$bal(S) = \bigcap_{|\lambda|\le 1} \lambda S.$$

In the context of a topological vector space (X,τ), we have some natural bornologies.

Examples 2.2 *Let (X,τ) be an $t.v.s.$, then the following collections of subsets of X are vector bornologies.*

1. The F-bornology β_F is the collection of all the τ-bounded subsets of X.
2. The H-bornology β_H is the collection of all compact subsets of X.
3. The G-bornology β_G is the collection of all finite subsets of X.

From the definitions of these bornologies, it is easy to see that

$$\beta_G \subseteq \beta_H \subseteq \beta_F.$$

For a proof of this result see [(2)].

In the sequel, X and Y denote Banach spaces, Ω is an open subset of X and $\mathcal{L}(X,Y)$ denotes the space of linear transformation from X to Y that are continuous.

Definition 2.3 *Let β be a bornology in X. We say that a function $f:\Omega \to Y$ is β-differentiable in $a \in \Omega$ if there exists a function $u \in \mathcal{L}(X,Y)$ such that for every $S \in \beta$*

$$u(h) = \lim_{t\to 0} \frac{f(a+th)-f(a)}{t} = 0 \ \ uniformly\ in\ \ h \in S.$$

The linear and continuous function u is called β-derivative of f at the point $a \in \Omega$ and is denoted by $u := d_\beta f(x)$.

The following result provides a differentiability criterium with respect to a given bornology:

Theorem 2.4 *Let X and Y be $t.v.s$, Ω an open subset of X, and β be a bornology in X. A necessary condition for $f:\Omega \to Y$ to be β-differentiable in $a \in \Omega$ is that for every $S \in \beta$*

$$\lim_{t\to 0^+} \frac{f(a+th)+f(a-th)-2f(a)}{t} = 0 \ \ uniformly\ in\ \ h \in S.$$

This result follows doing a suitable modification in the proof of proposition 1.23 in [(3)].

In the case of a convex continuous functions $f:\Omega \subset X \to Y$, we have the following characterization of the β-differentiable in $a \in \Omega$:

Theorem 2.5 *Let X be an $t.v.s.$, Ω an open convex subset of X, and β be a bornology* *in X. A necessary and sufficient condition for a convex and continuous function $f:\Omega \to \mathbb{R}$ be β-differentiable in $a \in \Omega$ is that for every $S \in \beta$*

$$\lim_{t\to 0^+} \frac{f(a+th)+f(a-th)-2f(a)}{t} = 0 \ uniformly\ in\ \ h \in S.$$

Similar considerations as theorem 2.4.

2.1 Single-Valued Maps Derivatives

Now, we will study the main notions of differentiation in a $t.v.s.$ linked to the bornologies mentioned before: the Frechet derivative linked to the bornology of the bounded subsets of a $t.v.s.$, the derivative of Hadamard linked to the bornology of the compact subsets and the derivative of Gâteaux linked to the bornology of the sets finite. Our interest in this work is to develop this theory in the case of Banach spaces.

Definition 2.1.1 *Let X and Y be Banach spaces, $\Omega \subset X$ be an open subset and $f:\Omega \to Y$ be a function.*

We say that the function $f:\Omega \to Y$ is *G-differentiable (differentiable in the sense of Gâteaux) at the point $a \in \Omega$ if there exists $u \in \mathcal{L}(X,Y)$ such that*

$$u(h) = \lim_{t\to 0} \frac{f(a+th)-f(a)}{t}, for\ every\ \ h \in X$$

and this limit is uniform on the finite subsets of X. In this case, we say that u is the G-derivative of f in a and $d_G f(a)(h) := \langle d_G f(a), h\rangle := u(h)$.

We say that the function $f:\Omega \to Y$ is H-differentiable (differentiable in the sense of Hadamard) at the point $a \in \Omega$ if there exists $u \in \mathcal{L}(X,Y)$ such that

$$u(h) = \lim_{t\to 0} \frac{f(a+th)-f(a)}{t}, for\ every\ \ h \in X$$

and this limit is uniform on the compact subsets of X. In this case, we say that u is the H-derivative of f in a and $d_H f(a)(h) := \langle d_H f(a), h\rangle := u(h)$.

We say that the function $f:\Omega \to Y$ is F-differentiable (differentiable in the sense of Fréchet) at the point $a \in \Omega$ if there exists $u \in \mathcal{L}(X,Y)$ such that

$$u(h) = \lim_{t\to 0} \frac{f(a+th)-f(a)}{t}, for\ every\ \ h \in X$$

and this limit is uniform on the bounded subsets of X. In this case, we say that u is the F-derivative of f in a and $d_F f(a)(h) := \langle d_F f(a), h\rangle := u(h)$.

When differentiability holds for any $a \in \Omega$, we say that f is *G-differentiable* in Ω (resp. *H-differentiable*) (resp. *F-differentiable*).

It is straightforward to see that the condition for uniform convergence is expressed as follows:

for all bounded set B of X and every $\varepsilon > 0$ there is a $\delta(B,\varepsilon) > 0$ such that

$$\left\| \frac{f(a+th)-f(a)}{t} - d_F f(a)(h) \right\|_Y < \varepsilon, \ if \ |t| < \delta \ forany \ h \in B. \qquad (1)$$

It is clear that if f is F-differentiable in $a \in \Omega$ with derivative u, then it is G-differentiable in a with derivative u.

The following elementary theorem is fundamental to the study of differentiability of convex functions:

Theorem 2.1.2 *Let X be a normed space. A convex continuous function defined on an open convex set with values in ℝ is necessarily locally Lipschitz.*

For a proof of this theorem see the work by R. Phelps in [3].

3 USCO Functions

In this section we will go deeper into the study of the class of usco functions. We start giving some basic definitions.

Definition 3.1 *(Multivalued function) Let X and Y be topological spaces. A multivalued function of a set X in a set Y is a correspondence $x \mapsto f(x)$, which assigns to each $x \in X$ a subset $f(x)$ of the set Y. It is possible that the set $f(x)$ is the empty set. The effective domain of this function is the set of the $x \in X$ such that $f(x) \neq \emptyset$.*

Definition 3.2 *(Upper/Lower semicontinuous multivalued function) Let X and Y be topological spaces. A multivalued function $f: X \to Y$ is upper semicontinuous in $a \in X$ if for every open set W in Y such that $W \supseteq f(a)$ (open neighborhood of $f(a)$) there exists an open set V of X such that $a \in V$ (neighborhood of a) and $f(x) \subseteq W$ for all $x \in V$. In the case that f is upper semicontinuous for any $a \in X$, we say that f is upper semicontinuous in X.*

A multivalued function $f: X \to Y$ is lower semicontinuous in $a \in X$ if for every open set W in Y such that $f(a) \cap W \neq \emptyset$, there exists an open set V of X such that $a \in V$ (neighborhood of a) and $f(x) \cap W \neq \emptyset$ for all $x \in V$. In the case that f is lower semicontinuous for any $a \in X$, we say that f is lower semicontinuous in X.

The first remark is that f is *continuous* in $a \in X$ if it is both upper semicontinuous and lower semicontinuous at the point a.

Definition 3.3 *(Graph of a multivalued function) Let X and Y be topological spaces. The graph of a multivalued function $f: X \to Y$ is the subset $Graph(f)$ of $X \times Y$ of the pairs $(x,y) \in X \times Y$ such that $y \in f(x)$.*

Definition 3.4 *The limit values of a net $(y_t)_{t \in T}$ in a topological space (Y,α) are the* *elements of the set*

$$\bigcap_{t\in T}\overline{\{y_s : s \succcurlyeq t\}}$$

where $\succcurlyeq$ is the partial order relation in T.

We note that y is a limit value of the indicated net if and only if there exists a subnet that converges to y.

Theorem 3.5 *Let X and Y be topological spaces. A multivalued function $f: X \to Y$ is upper semicontinuous if and only if for any closed set $C \subset Y$, the set*

$$\{x \in X: f(x) \cap C \neq \emptyset\}$$

is closed on X.

Proof. Suppose that f is upper semicontinuous, we want to show that

$$\{x \in X: f(x) \cap C \neq \emptyset\} \text{ is closed on } X.$$

Let C be a closed subset of Y and $A = C^c$, then A is an open set in Y and if $f(x) \cap C = \emptyset$, then $f(x) \subseteq A$. Since f is upper semicontinuous, there is a neighborhood open V of x such that $f(z) \subseteq A$ for all $z \in V$. But this means that $f(z) \cap C = \emptyset$ for all $z \in V$ and, therefore,

$$(\{x \in X: f(x) \cap C \neq \emptyset\})^c$$

is open set and so $\{x \in X: f(x) \cap C \neq \emptyset\}$ is closed on X. Suppose now that

$$\{x \in X: f(x) \cap C \neq \emptyset\}$$

is closed with C is closed, we want to establish that f is upper semicontinuous. Let $a \in X$ and W is an open subset of Y such that $f(a) \subseteq W$. Then, W^c is a closed set and by hypothesis

$$\{x \in X: f(x) \cap W^c \neq \emptyset\}$$

is closed in X and its complement V is an open set. Now, $a \in V$ since $f(a) \cap W^c = \emptyset$ and $f(x) \cap W^c = \emptyset$ for all $x \in V$, so that $f(x) \subseteq W$ for all $x \in V$.

Observe that now we are able to introduce a relation of order on the set of multivalued functions $\mathcal{T}(X,Y)$ with X and Y being topological spaces. For f, g in $\mathcal{T}(X,Y)$, se define the order $\preccurlyeq$ as

$$f \preccurlyeq g, \qquad if\ \ f(x) \subseteq g(x)\ \ for\ any\ \ x \in X.$$

Definition 3.6 *(Usco Function) Let X and Y be topological spaces. A multivalued function $f: X \to Y$ is a usco function if f is upper semicontinuous such that $f(x) \neq \emptyset$ and is compact for any $x \in X$. We denote the set of usco functions from X to Y by $\mathcal{U}(X,Y)$.*

We say that $f \in \mathcal{U}(X,Y)$ is a minimal usco function if f is a minimal element of the ordered set $(\mathcal{U}(X,Y), \preccurlyeq)$. This means that, if $g \in \mathcal{U}(X,Y)$ and $g \preccurlyeq f$, then $f = g$.

Now, we will establish some results about usco functions.

Theorem 3.7 *For every* $u \in \mathcal{U}(X,Y)$*, there exists a minimal usco function* $f \in \mathcal{U}(X,Y)$ *such that* $f \preccurlyeq u$.

Proof. Let $u \in \mathcal{U}(X,Y)$ and let $\mathcal{H}$ be the collection of usco functions h such that $h \preccurlyeq u$. Let us show that every chain $\mathcal{L}$ contained in $\mathcal{H}$ is bounded below. If $\mathcal{F}$ is a finite subset in $\mathcal{L}$, then $\bigcap_{f\in\mathcal{F}} f(x)$ is a closed and nonempty subset contained in the compact set $u(x)$ and which we can order $\mathcal{F}$ linearly. Then $\bigcap_{h\in\mathcal{L}} h(x)$ is a closed and nonempty set contained in the compact set $u(x)$ and is therefore compact. We apply theorem 3.5 to the chain $\{h: h \in \mathcal{L}\}$ to conclude that the function

$$x \mapsto g(x) = \bigcap_{h\in\mathcal{L}} h(x)$$

is upper semicontinuous and thus, a usco function that minorizes $\mathcal{L}$. By Zorn's lemma $\mathcal{H}$ has a minimal element.

Lemma 3.8 *Let* (X,α) *and* (Y,τ) *be topological spaces and* $f: X \to Y$ *be a usco function. If* $(x_t, y_t)_{t\in T}$ *is a net in* $Graph(f)$ *and* $x_t \to x$*, then* $(y_t)_{t\in T}$ *has at least one limit value in* $f(x)$.

Proof. By contradiction, we assuming that the net $(y_t)_{t\in T}$ has not limit values in $Graph(f)$. That is, for all $z \in f(x)$, there exists a t_z such that $z \notin \overline{\{y_s: s \succcurlyeq t_z\}}$. Let W_z be an open neighborhood of z such that $W_z \cap \overline{\{y_s: s \succcurlyeq t_z\}} = \emptyset$. Since $f(x)$ is compact, there exists a finite subset $F \subseteq f(x)$ such that $f(x) \subseteq \bigcup_{z\in F} W_z = A$. For the upper semicontinuity of f, there exists an open neighborhood $V \subseteq X$ of x such that

$$f(b) \subseteq W \ \ for\ any\ \ b \in V.$$

As $x_t \to x$, there exists a $\bar{t} \in T$ (where T is a directed set) such that $x_t \in V$ provided that $t \succcurlyeq \bar{t}$. We can suppose that $t \succcurlyeq \bar{t} \succcurlyeq \sup\{t_z: z \in F\}$. Consequently, $f(x_t) \subseteq A$ if $t \succcurlyeq \bar{t}$. By hypothesis, we have $(x_t, y_t) \in Graph(f)$, i.e., $y_t \in A$ and $y_t \in W_z$ for some $z \in F$, which implies that $W_z \cap \overline{\{y_s: s \succcurlyeq t_z\}} \neq \emptyset$, which is a contradiction.

Theorem 3.9 *(Characterization of usco functions) Let* (X,α) *and* (Y,β) *be topological spaces. A multivalued function* $f: X \to Y$ *is a usco function if and only if its graph* $Graph(f)$ *is a closed set. Moreover, there exists a usco function* $u \in \mathcal{U}(X,Y)$ *such that* $f \preccurlyeq u$.

Proof. Assume that f is a usco function. If prove that $(x,y) \in \overline{Graph(f)}$, then we will show that $(x,y) \in Graph(f)$, i.e., $y \in f(x)$. There is a net $(x_t, y_t)_{t\in T}$ such that $x_t \to x$, $y_t \to y$ and $y_t \in f(x_t)$. By lemma 3.8, $(y_t)_{t\in T}$ has at least one adhesion value in $f(x)$. That is, there is a subnet $(y_{t_s})_{s\in S}$ such that $y_{t_s} \to y' \in f(x)$ and by the uniqueness of the limits, $y = y' \in f(x)$. It is clear that $f \preccurlyeq u = f$.

In the other direction, if $Graph(f)$ is closed and there is a usco function u such that $f \preccurlyeq u$, then f is a usco function. By theorem 3.5, it suffices to show that $f(x)$ is closed for all $x \in X$. Let $y \in \overline{f(x)}$ and we will show that $y \in f(x)$. Then there is a net $(y_t)_{t\in T}$ in $f(x)$ such that $y_t \to y$. Let $x_t = x$ for all $t \in T$. Then $(x_t, y_t)_{t\in T}$ is a net in $Graph(f)$ that fulfills the conditions of lemma 3.8 and, therefore, a subnet $(y_t)_{t\in T}$ converge to $y' \in f(x)$ and as $y_t \to y$, it follows that $y' = y \in f(x)$.

The following lemma will be helpful for the main result in this section.

Lemma 3.10 *Let (X,τ) be the topological space, (Y,σ) Hausdorff space and $f: X \to Y$ a usco function. The following are equivalent:*

(i). *f is a minimal usco function.*
(ii). *If A is an open subset of X, W is an open subset of Y and $f(a) \cap W \neq \emptyset$ for some $a \in A$, then there exists an open nonempty subset $V \subseteq A$ such that $f(x) \subseteq W$ for all $x \in V$.*
(iii). *If A is an open subset of X and C is a closed subset of Y such that $f(a) \cap C \neq \emptyset$ for all $a \in A$, then $f(a) \subseteq C$ for all $a \in A$.*

Proof. We will show that $(i) \Rightarrow (ii)$. Let $A \subseteq X$ and $W \subseteq Y$ open subsets as in (ii). We need to establish that exists an $a_0 \in A$ such that $f(a_0) \subseteq W$ because the fact that f is upper semicontinuous implies that there is an open neighborhood V in A of a_0 such that

$$f(x) \subseteq W \ \ for\ all\ \ x \in V.$$

We argue by contradiction, let us assume that this statement is false and let $C := W^c$ closed in Y. Then, $f(x) \cap C \neq \emptyset$ for any $x \in A$. We define the function $h: X \to Y$ by

$$h(x) = \begin{cases} f(x) \cap C & if\ x \in A \\ f(x) & if\ x \in A^c. \end{cases}$$

It is clear that $h(x) \neq \emptyset$, closed for all $x \in X$ and $h \preccurlyeq f$. Then, we conclude that $h \in \mathcal{U}(X,Y)$ by theorem 3.9. Since f is a minimal usco function by hypothesis, then $h = f$ and consequently, $h(x) = f(x) \subseteq C$ for all $x \in A$. This is a contradiction, since by hypothesis $f(a) \cap W \neq \emptyset$ for some $a \in A$.

Now we show that $(ii) \Rightarrow (i)$. By hypothesis $f: X \to Y$ is a usco function. From theorem 3.7, there exists a minimal usco function $g \preccurlyeq f$ and $g = f$, as we will show below. If f and g are not equal, there exists $x_0 \in X$ such that $g(x_0) \neq f(x_0)$. As $g(x_0) \subset f(x_0)$ there exists $z \in f(x_0)$ such that $z \notin g(x_0)$. Thus, there exists an open T in Y such that $z \in T$ and $g(x_0) \cap \overline{T} = \emptyset$ since $g(x_0)$ is compact. From this, it follows that $g(x_0) \subseteq \left(\overline{T}\right)^c$. Let $W := \left(\overline{T}\right)^c \subseteq Y$ be an open neighborhood of $g(x_0)$. By the upper semicontinuity of g, there exists an open neighborhood $A \subseteq X$ of x_0 such that

$$g(x) \subseteq W \ \ for\ any\ \ x \in A.$$

This construction allows us to see that $x_0 \in A$ and $f(x_0) \cap T \neq \emptyset$. Then, there is a subset open nonempty $V \subseteq A$ such that

$$f(x) \subseteq T \;\; for\; any\;\; x \in V,$$

which is a contradiction, since $g(x) \subseteq f(x) \subseteq T$, $g(x) \subseteq W$ and $T \cap W = \emptyset$ if $x \in V$.

Now we see that $(ii) \Rightarrow (iii)$. Let A be a subset open in X and C be a subset closed in Y as in (iii). Suppose there exists $a \in A$ such that $f(a)$ is not contained in C. Then, $f(a) \cap W \neq \emptyset$ for some $a \in A$, where $W := C^c$ is an open subset in Y. For (ii) there exists a nonempty open subset $V \subseteq A$ such that $f(x) \subseteq W$ for all $x \in V$, i.e., $f(V) \cap C = \emptyset$, which is a contradiction, since by hypothesis $f(x) \cap C \neq \emptyset$ for all $x \in A$.

We will show that $(iii) \Rightarrow (ii)$. Let $A \subseteq X$ and $W \subseteq Y$, with $f(a_0) \cap W \neq \emptyset$ such that $a_0 \in A$. Let us prove that there exists $a \in A$ such that $f(a) \subseteq W$. Otherwise, we would have $f(a) \cap W^c \neq \emptyset$ for all $a \in A$, where $C := W^c$ is a closed subset in Y. Then, for (iii) we have $f(a) \subseteq W^c$ for all $a \in A$ and hence $f(A) \cap W = \emptyset$, which is a contradiction. Therefore, W is an open neighborhood in Y of $f(a_0)$ and by the upper semicontinuity of f, there exists an open neighborhood $V \subseteq X$ of a_0 contained in A such that $f(V) \subseteq W$. In other words, $f(x) \subseteq W$ for all $x \in V$.

We also have the following theorem of minimal usco functions:

Theorem 3.11 *(Characterization of minimal usco functions) Let (X,τ) and (Y,σ) be topological spaces and $f: X \to Y$ be a usco function. Then the following statements are equivalent:*

1) *f is a minimal usco function.*
2) *For every topological space (Z,α) and any continuous single-valued function $g: Y \to Z$, $g \circ f$ is a minimal usco function of X in Z.*

Proof. We will show 1) $\Rightarrow$ 2).

(i) $(g \circ f)(x)$ is compact and nonempty for all $x \in X$. Indeed, as f is a usco function, then $f(x)$ is compact for all $x \in X$. Then $g(f(x))$ is compact since g is continuous.

(ii) $g \circ f$ is upper semicontinuous. In fact, let $x_0 \in X$ and W be an open neighborhood in Z such that $g(f(x_0)) \subseteq W$. By continuity, we have $A := g^{-1}(W)$ is open in Y such that $f(x_0) \subseteq A$. Since f is upper semicontinuous, there is a neighborhood open V in X of x_0 such that $f(x) \subseteq A$ for all $x \in V$ and, therefore, $g(f(x)) \subseteq W$ for any $x \in V$. So, we have shown that $(g \circ f) \in \mathcal{U}(X,Z)$.

It remains to show that $g \circ f$ is a minimal usco function. Using equivalences established in lemma 3.10, we must verify that $g \circ f$ satisfies (ii). Let A be an open subset in X and W subset open in Z such that $((g \circ f)(a)) \cap W \neq \emptyset$ for some $a \in A$.

For continuity of g, we have $g^{-1}(W)$ is an open subset in Y such that $f(a) \subseteq g^{-1}(W)$ (this is because $\big((g \circ f)(A)\big) \cap W \neq \emptyset$ if and only if $f(A) \cap g^{-1}(W) \neq \emptyset$). Since f is upper semicontinuous, there is an open neighborhood $V \subseteq A$ of a such that

$$f(x) \subseteq g^{-1}(W) \;\; for\ any\ \ x \in V$$

and, therefore, $g\big(f(x)\big) \subseteq W$ for any $x \in V$.

We will show 2) $\Rightarrow$ 1). Let $Z = Y$ and $i_Y: Y \to Y$ be the identity function $i_Y(y) = y$.

We observe that $i_Y \circ f = f$, which by hypothesis is a minimal usco function.

Definition 3.12 *(Residual space) Let* (X, α) *be a topological space. We said that* $G \subseteq X$ *is a residual set, if there is a family of open sets* $(\mathcal{U}_n)_{n\in\mathbb{N}}$ *which are dense in* X *such that* $\bigcap_{n\in\mathbb{N}} \mathcal{U}_n \subseteq G$.

Recall that a subset D of X is *dense* if its closure $\overline{D}$ is equal to X.

Definition 3.13 *(Baire space) A Baire space is a topological space* (X, α) *with the property that any residual is dense in* X.

Regarding Baire space, we have the following well known examples (see [4]):

1. Any locally compact topological space is a Baire space.
2. Any open subset of a Baire space is a Baire space.
3. Any complete metric space is a Baire space.

Theorem 3.14 *Let* (X, α) *be a Baire space and* (Y, d) *be a complete metric space. Then any minimal usco function* $f: X \to Y$ *is single-valued in some residual subset* R *of* X.

Proof. Let $f: X \to Y$ be a minimal usco function, we will see that, for every $\varepsilon > 0$, there exists an open set V such that diameter of $f(V)$ is less than ε ($diam\big(f(V)\big) < \varepsilon$). In fact, let $y \in f(X)$ and $B\left(y, \frac{\varepsilon}{2}\right)$. There is at least one $a \in X$ such that $f(a) \cap B\left(y, \frac{\varepsilon}{2}\right) \neq \emptyset$, otherwise $f(x) \subseteq \left(B\left(y, \frac{\varepsilon}{2}\right)\right)^c$ for all $x \in X$. By the lemma 3.10.(ii), there exists an open nonempty subset V of X such that $f(x) \subseteq B\left(y, \frac{\varepsilon}{2}\right)$ for all $x \in V$ and consequently

$$diam\big(f(V)\big) \leq \frac{\varepsilon}{2}.$$

Let $A_\varepsilon = \bigcup \{V \subseteq X: V\ open\ and\ diam\big(f(V)\big) < \varepsilon\}$. It is clear that this set is open. We will show that A_ε is dense. Let $x \in X$ and U be an open neighborhood of x. If $y \in f(x)$ and $B\left(y, \frac{\varepsilon}{2}\right)$ then $f(x) \cap B\left(y, \frac{\varepsilon}{2}\right) \neq \emptyset$ and again by lemma 3.10.(ii), there is an open nonempty subset V of U such that $f(v) \subseteq B\left(y, \frac{\varepsilon}{2}\right)$ for all $v \in V$, i.e.,

$$diam(f(V)) \leq \frac{\varepsilon}{2}$$

and, therefore, $V \subseteq A_\varepsilon$. We have shown that $U \cap A_\varepsilon \neq \emptyset$. If $\varepsilon = n^{-1}$, we will write A_n instead of $A_{\frac{1}{n}}$. Then $\{A_n : n \in \mathbb{N}\}$ is a countable collection of open subsets which are dense in X. Since this is a Baire space, then $R := \bigcap_{n\in\mathbb{N}} A_n$ is a subset residual dense. If $x \in R$, then

$$diam(f(x)) < \frac{1}{n} \ for\ any\ n \in \mathbb{N},$$

and, therefore, $f(x)$ consists of only one element. That is, $f|_R$ is a single-valued function.

3.1 Usco Functions and Continuity of Subdifferential of a Convex Function

Definition 3.1.1 *(Dual space) Let X be a Banach space. The topological dual of X is defined as the Banach space* $X' = \mathcal{L}(X, \mathbb{R})$. *Hereafter we set the notation*

$$x'(x) := \langle x', x\rangle, \quad x' \in X', \quad x \in X.$$

If X is a Banach space, the weak- * *topology of the dual* X' denoted by $\sigma(X', X)$ is generated by the family $\{p_x : x \in X\}$ of seminorms, where $p_x(x') = |\langle x', x\rangle|$.

We note that $(X', \sigma(X', X))$ is a completely regular topological space. A fundamental weak neighborhood of $a \in X$ is any set of the form

$$B_{F',\varepsilon}(a) = \bigcap_{x'\in F'} \{x \in X : |\langle x', x - a\rangle| < \varepsilon\},$$

where F' is a finite subset of X' and $\varepsilon > 0$.

Definition 3.1.2 *(The subdifferential) Let Ω be a convex open subset of a Banach space X and let* $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ *be a convex and continuous function, and let* $x \in domf$. *The subdifferential of f in the point* $a \in \Omega$, *is the set of* $u \in X'$ *such that*

$$f(x) - f(a) \ \geq \ u(x - a) = u(x) - u(a) \ for\ any\ x \in \Omega.$$

We denote by $\partial f(a)$ *the subdifferential of f at point a. Any function* $x \to u_x \in X'$ *such that* $u_x \in \partial f(x)$ *for all* $x \in \Omega$ *is called subgradient of f at x. A function f is called subdifferentiable in* $a \in \Omega$, *if there is at least one subgradient in a. A function f is called subdifferentiable, if it is subdifferentiable at each* $x \in domf$.

Definition 3.1.3 *Let X be a Banach space and Ω be a convex open subset of X. A function* $f : \Omega \to \mathbb{R}$ *called is radially differentiable at* $a \in \Omega$ *in the direction h if there exists the limit*

$$h \mapsto \frac{f(a+th) - f(a)}{t}, \ when \ t \to 0^+. \tag{2}$$

The value of this limit is called the radial derivative of f at point a in the direction h. We denote

$$d^+f(a,h):=\lim_{t\to 0^+}\frac{f(a+th)-f(a)}{t} \tag{3}$$

the radial derivative of f at point a in the direction h. The function $d^+f(a,h)$ is well defined, it is sublineal, convex and continuous. If $d^+f(a,h)$ exists for all $h\in X$, then we say that f is radially differentiable at $a\in\Omega$.

We know that every convex real valued function defined in an open is radially differentiable if

$$d^+f(a,h)=\lim_{t\to 0^+}\frac{f(a+th)-f(a)}{t} \quad and \quad d^-f(a,h)=\lim_{t\to 0^-}\frac{f(a+th)-f(a)}{t} \tag{4}$$

then

$$d^-f(a,h)\ \le\ d^+f(a,h), \qquad for\ any\ \ a\in\Omega\ \ and\ any\ \ h\in X.$$

The radial derivative $d^{\pm}f(a,h)$ it allows us to locate the subgradient at the point $a\in\Omega$ in the following sense:

Lemma 3.1.4 *Let X be a Banach space, Ω be a convex open subset of X, $f:\Omega\to\mathbb{R}$ be a convex and continuous function. Then $u\in\partial f(a)$ if and only if*

$$d^-f(a,h)\ \le\ u(h)\ \le\ d^+f(a,h), \qquad for\ all\ \ a\in\Omega\ \ and\ all\ \ h\in X.$$

Proof. If $u\in\partial f(a)$, then $u:X\to\mathbb{R}$ is linear and continuous. Also if $a+th\in\Omega$ for t small enough and

$$u(th)=tu(h)=u(a+th)-u(a)\ \le\ f(a+th)-f(a).$$

Dividing by $t>0$ and taking $t\to 0^+$, we have $u(h)\le d^+f(a,h)$. On the other hand,

$$\frac{f(a+th)-f(a)}{t}=\frac{f\big(a+(-t)(-h)\big)-f(a)}{-t}(-1)$$

and, therefore, when t tends to zero by the left, we obtain that

$$d^-f(a,h)=-d^+f(a,-h).$$

Consequently,

$$u(-h)=-u(h)\ \le\ d^+f(a,-h)=-d^-f(a,h)$$

and, therefore, $d^-f(a,h)\le u(h)$.

Remark 3.1.5 *For a Banach space X, we consider in X' the weak-* topology $\sigma(X',X)$. If $f:X\to\mathbb{R}$ is a convex and continuous function, we will show that $\partial f:X\to X'$ is a usco function, considering in X the norm topology and in X' the weak-* topology.*

Lemma 3.1.6 *Let X be a normed space, let Ω be an open and convex subset and $f: \Omega \to \mathbb{R}$ be a convex and continuous function. Then,*

1. *The subdifferential function is a locally bounded function multivalued. In other words, for all $a \in \Omega$, exists $r > 0$ and $m > 0$ such that*

$$B(a,r) \subseteq \Omega \text{ and } \|x'\| \leq m \text{ for any } x' \in \partial f(x) \text{ and any } x \in B(a,r).$$

2. *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in Ω that converges to $x \in \Omega$ and $(v_n)_{n\in\mathbb{N}}$ is a sequence in X' such that $v_n \in \partial f(x_n)$ for all $n \in \mathbb{N}$, then $(v_n)_{n\in\mathbb{N}}$ is bounded and the set of the $\sigma(X',X)$-adherence values is nonempty and is contained in $\partial f(x)$.*

For a proof of this lemma see the work by R Phelps [(3)].

Theorem 3.1.7 *Let X be a normed space, let Ω be an open and convex subset. If $f: \Omega \to \mathbb{R}$ is a convex and continuous function, then $x \mapsto \partial f(x)$ of X in X' is a usco function, considering in Ω the norm topology and in X' the weak-* topology $\sigma(X',X)$.*
Proof. We must show that:

i. $\partial f(x)$ is $\sigma(X',X)$-compact and $\partial f(x) \neq \emptyset$ for all $x \in \Omega$.
ii. ∂f is upper semicontinuous.

Let us show (i). Since f is continuous at $a \in \Omega$ and convex, then $epi(f) \subseteq X \times \mathbb{R}$ is a convex set such that $int(epi(f)) \neq \emptyset$. In particular if $(a, f(a)) \notin int(epi(f))$, then under the Hahn-Banach theorem, it exists $u \in X'$ which separates (but not strictly) the point $(a, f(a))$ of the convex set. We can observe that $u \in X'$ is the subgradient of f at a and, therefore, $\partial f(x) \neq \emptyset$ for all $x \in \Omega$. Now, since $\partial f(x) \subseteq X'$ is bounded, by the Banach-Alaoglu theorem, this set is $\sigma(X',X)$-relatively compact. Now, to conclude the proof we need to prove that $\sigma(X',X)$-closed. In addition, $\partial f(x)$ is $\sigma(X',X)$-compact. We recall that, $u \in \partial f(x)$ if and only if $u(y) \leq d^+ f(x,y)$ for $y \in X$ (by lemma 3.1.4) and

$$\partial f(x) = \bigcap_{y\in X} \{u \in X' : u(y) \leq d^+ f(x,y)\}$$

is $\sigma(X',X)$-closed since for each $y \in X$, the function $u \mapsto u(y)$ is $\sigma(X',X)$-continuous. Let us show (ii). We will argue by contradiction, assuming that $x \mapsto \partial f(x)$ is not upper semicontinuous in a $a \in \Omega$. This means that there exists a $\sigma(X',X)$-neighborhood open W of $\partial f(a)$, a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω and a sequence $(v_n)_{n\in\mathbb{N}}$ in X' such that $v_n \in \partial f(x_n)$ and $v_n \notin W$ for all $n \in \mathbb{N}$. By the lemma 3.1.6.2, the sequence $(v_n)_{n\in\mathbb{N}}$ has at least one $\sigma(X',X)$-adherence value $u \in \partial f(a)$. As W is a $\sigma(X',X)$-neighborhood of u, then $v_n \in W$, which is a contradiction.

Theorem 3.1.8 *Let X be a Banach space, Ω an open and convex subset and $f: \Omega \to \mathbb{R}$ is a convex and continuous function. If f is F-differentiable in $a \in \Omega$, then the* *respect to the norm topologies.*

For a proof of this theorem see [3].

Lemma 3.1.9 *Let X be a normed space, Ω an open and convex subset and $f\colon \Omega \to \mathbb{R}$ a continuous convex function. A sufficient condition for that f to be F-differentiable in $a \in \Omega$ is that there is a continuous selection in a with respect to the norm topologies.*

For a proof of this lemma see [3].

4 The Asplund Spaces

We recall that a subset G of a topological space (X, α) is a *set* G_δ if it can be expressed as a countable intersection of open and dense subsets. In complete metric spaces and so, in Banach spaces, any residual subset is dense by the Baire's theorem.

Definition 4.1 *(Asplund F-space/$\mathcal{G}$-space/$\mathcal{GD}$-space) An Asplund F-space is a Banach space with the following property: If f is a continuous convex function defined in open and convex subset $U \subset X$, then F-differentiable in a dense subset G_δ of U.*

A $\mathcal{G}$-Asplund is a Banach space with the following property: If f is a continuous convex function defined in open and convex subset $U \subset X$, then G-differentiable in a dense subset G_δ of U.

An Asplund $\mathcal{GD}$-space is a Banach space with the following property: If f is a continuous convex function defined in open subset $U \subset X$, then G-differentiable in a dense subset G_δ of U.

We note that a $\mathcal{G}$-Asplund space is also an Asplund $\mathcal{GD}$-space. We want to point out that in the usual literature, the Asplund F-spaces are called the *Asplund spaces*, the $\mathcal{G}$-Asplund are called *weak Asplund spaces* and Asplund $\mathcal{GD}$-spaces are called *weak differenciable spaces*. We could talk about the H-*Asplund* spaces, but it does not make much sense because it has been shown that continuous convex function wich are G-differentiables are H-differentiables. In the original article by E. Asplund [5], a $\mathcal{G}$-Asplund is a *weak differentiability spaces* (WDS) and a Asplund F-space is called *strong differentiability spaces* (SDS). The definition we use comes from R. Phelps [6].

We will denote by $\mathfrak{U}_G$ the class of the $\mathcal{G}$-Asplund and $\mathfrak{U}_F$ the class of Asplund F-spaces.

4.1 Asplund F-Spaces and Dentables Subset of a Banach Space

In this section we present the proof of the Namioka-Phelps theorem, which characterizes Asplund F-spaces by means of the geometric condition of dentability. First, we discuss the notion of the dentable set. To do that, if M is a subset of a Banach space X, we define the *support function* of this set M as follows:

$$x' \mapsto p_M(x') := \sup_{x \in M} \langle x, x' \rangle$$

of X' in $\mathbb{R} \cup \{\infty\}$. If we replace M by $T = adh_{\sigma(X',X)} \circ cv(M)$, then T is convex, closed and bounded in the norm topology and $p_M = p_T$. This function is sublineal and lower semicontinuous considering in X' the weak- * topology denoted by $\sigma(X', X)$ because it is the upper lower bound of lower semicontinuous functions.

Suppose now that X is a Banach space and M' is a subset of X'. Proceeding as before, we define the *support function* of the set M' as follows:

$$x \mapsto p_{M'}(x) := \sup_{x' \in M'} \langle x', x \rangle$$

of X in $\mathbb{R} \cup \{\infty\}$. If $T' = adh_{\sigma(X,X')} \circ cv(M')$, then T' is convex, $\sigma(X, X')$-closed (topology in X) and bounded in the norm topology and $p_{M'} = p_{T'}$.

Definition 4.1.1 *(X-slice) Let X be a Banach space, $\alpha > 0$ and $x \in X$. For a nonempty bounded subset M of X, we define the (α, x')-slice of M as the set*

$$\mathcal{R}(\alpha, x'; M) := \{x \in M : \langle x, x' \rangle > p_M(x') - \alpha\}.$$

Is clear that every slice is a relatively open set of M with the topology $\sigma(X, X')$, i.e., $\sigma(X, X')|_M$ (restriction of $\sigma(X, X')$ to M).

Definition 4.1.2 *(X'-slice) Let X be a Banach space, $\alpha > 0$ and $x \in X$. For a nonempty bounded subset M' of X', we define the weak *- (α, x)-slice of M' as the set*

$$\mathcal{R}'(\alpha, x; M') := \{x' \in M' : \langle x', x \rangle > p_{M'}(x) - \alpha\}.$$

We note that $\mathcal{R}'(\alpha, x; M')$ is open in the topology $\sigma(X', X)|_{M'}$. On the other hand, if $\alpha < \beta$, then

$$\mathcal{R}'(\alpha, x; M') \subseteq \mathcal{R}'(\beta, x; M').$$

Moreover, if $T' = adh_{\sigma(X,X')} \circ cv(M')$, then $\mathcal{R}'(\alpha, x; M') \subseteq \mathcal{R}'(\alpha, x; T')$ since $p_{M'} = p_{T'}$.

Definition 4.1.3 *(Dentable set) Let X be a Banach space. A subset $M \subset X$ is called dentable if it admits slices of arbitrarily small diameter. We say that a subset $M' \subset X'$ is weak *-dentable if it admits weak *-(α, x)- slice of arbitrarily small diameter. A Banach space is dentable if any bounded subset is dentable.*

Theorem 4.1.4 *(Namioka - Phelps). A Banach space X is an Asplund F-space if and only if its dual X' is weak *- dentable.*

Proof. Suppose X is an Asplund F-space and show that all bounded subset of X' admits slices of arbitrarily small diameter, otherwise, out so there would be a bounded subset M' of X' in which all (α, x)-slice has a diameter $r > 0$.

Let $T' = adh_{\sigma(X',X)} \circ cv(M')$. This set is convex, $\sigma(X',X)$-closed and bounded. By the Banach-Alaoglu theorem, T' is $\sigma(X',X)$-compact. Since $M' \subseteq T'$, all (α, x)-slice of T' has a diameter $\geq r$, we will show that the continuous sublinear function

$$x' \mapsto p(x) = p_{M'}(x) := \sup_{x' \in M'} \langle x', x \rangle$$

of X in $\mathbb{R}$ is not F-differentiable in any point, which contradicts that X is an Asplund F-space. With this purpose in mind, let $\varepsilon = \frac{r}{4}$ and $\delta > 0$ arbitrary. We will choose α and β appropriately. For now, we assume that $r - 2\beta > 0$, $\alpha > 0$ and $\beta > 0$. If $x \in X$, then $diam\mathcal{R}'(\alpha, x; T') \geq r$, so that there are a', $b' \in \mathcal{R}'(\alpha, x; T')$ such that $\|a' - b'\|_{X'} \geq r - \beta$,

$$\langle a', x \rangle \ > \ p(x) - \alpha \quad and \quad \langle b', x \rangle \ > \ p(x) - \alpha.$$

There exists a $y \in X$ such that $\|y\|_X = 1$ and $\langle a' - b', y \rangle > r - 2\beta$. Now, if $t > 0$, then

$$\begin{aligned} p(x + ty) + p(x - ty) - 2p(x) \ &\geq \ \langle a', x + ty \rangle + \langle b', x - ty \rangle - \langle a' + b', x \rangle - 2\alpha \\ &= \ t\langle a' - b', y \rangle - 2\alpha \ \geq \ t(r - 2\beta) - 2\alpha. \end{aligned}$$

We can now choose α and β appropriately. Let $\beta = \frac{r}{4}$ and $\alpha = \frac{r\delta}{4}$. So

$$\frac{p(x+ty)+p(x-ty)-2p(x)}{t} \ \geq \ (r - 2\beta) - \frac{2\alpha}{t} = \frac{r}{2} - \frac{r\delta}{8t}. \tag{5}$$

Since inequality (5) holds for all $t > 0$, in particular if $t = \frac{\delta}{2}$, so we have shown that if $\varepsilon = \frac{r}{4}$ and δ is arbitrary, then there is always a t and a y such that

$$0 < t < \delta, \qquad \|y\|_X = 1 \quad and \quad \frac{p(x + ty) + p(x - ty) - 2p(x)}{t} \ \geq \ \frac{r}{4}.$$

In virtue of theorem 2.4 with the bornology β_F, we conclude that p is not F-differentiable in x, which is a contradiction.

Now, we will show that X is an Asplund F-space. Let Ω be an open subset of X and $f: \Omega \to \mathbb{R}$ a convex function. Theorem 2.5 makes possible to locate the points of differentiability of f. The bornology that we are going to consider is β_F, of all bounded subsets X since here the F-derivative is defined. Let $D(f, \varepsilon)$ be the set of $x \in \Omega$ for which there is a $\delta(x, \varepsilon) > 0$ such that

$$\frac{f(x + th) + f(x - th) - 2f(x)}{t} \ < \ \varepsilon \ if \ \ 0 < t < \delta \ \ for\, any \ \ h \in X \ \ with \ \ \|h\| \ \leq \ 1.$$

Then:

(i). $D(f, \varepsilon)$ is open for all $\varepsilon > 0$.

(ii). $\bigcap_{\varepsilon > 0} D(f, \varepsilon)$ is the set of F-differentiability of function f.

The proof of (ii) is obvious. We will proof (i), i.e., $D(f,\varepsilon)$ is open. Since f is locally Lipschitzian by theorem 2.1.2, there exists a $r>0$ and a constant $k>0$ such that $B(x,r)\subseteq\Omega$ and

$$a,b\in B(x,r)\quad implies\quad |f(b)-f(a)|\ \leq\ k\|b-a\|_X.$$

By the definition of $D(f,\varepsilon)$, there exists $\delta>0$ such that $\delta<\frac{r}{2}$ and

$$\frac{f(x+th)+f(x-th)-2f(x)}{t}\ \leq\ \eta\ <\ \varepsilon\quad if\quad \|h\|_X\leq 1\quad and\quad 0<t<\delta.$$

Let $z\in B\left(x,\frac{r}{2}\right)$. Then $z+th,\ z-th\in B(x,r)$ since

$$\|z\pm th-x\|_X\ \leq\ \frac{r}{2}+t\|h\|_X\ <\ \frac{r}{2}+t\ <\ \frac{r}{2}+\delta\ <\ r.$$

Now let us observe that if $t=\delta$, then

$$\begin{aligned}\frac{f(z+\delta h)+f(z-\delta h)-2f(z)}{\delta}&\\ &\leq\ \frac{f(x+\delta h)+f(x-\delta h)-2f(x)}{\delta}+\frac{|f(z+\delta h)-f(x+\delta h)|}{\delta}\\ &+\frac{|f(z-\delta h)-f(x-\delta h)|}{\delta}+\frac{2|f(z)-f(x)|}{\delta}\\ &\leq\ \eta+\frac{4k}{\delta}(\|z-x\|_X).\end{aligned}$$

Let $\alpha=\min\left\{(\delta/4k)(\varepsilon-\eta),\frac{r}{2}\right\}$. Consequently,

$$\frac{f(z+th)+f(z-th)-2f(z)}{t}\ <\ \varepsilon,\qquad if\ \|z-x\|_X<\alpha,$$
$$\|h\|_X\leq 1\ and\ 0<t<\delta.$$

Once the points of F-differentiability of f are located, it remains to show that $A_\varepsilon:=D(f,\varepsilon)$ is dense for every $\varepsilon>0$. On the one hand, we have that Ω is a Baire space. If $\varepsilon=n^{-1}$, we will write $A_n=D\left(f,\frac{1}{n}\right)$ instead of $A_{\frac{1}{n}}$. Then $\{A_n:n\in\mathbb{N}\}$ is a countable collection of open subsets in Ω. Since Ω is a Baire space, if we define

$$R:=\bigcap_{n\in\mathbb{N}}A_n$$

(set of differentiability of f) is a set G_δ.

We will show that $D(f,\varepsilon)$ is dense for every $\varepsilon>0$. That is, if $x_0\in\Omega$ (fixed but arbitrary) any open neighborhood W of x_0 intersects $D(f,\varepsilon)$. As the subdifferential function is locally bounded, we can suppose that $\partial f(W) = T'$ is a bounded subset of X'. As X' is weak *-dentable, there exists a weak-* (α, z)-slice of T' of arbitrarily small diameter, let say $< \varepsilon$. If $a' \in \mathcal{R}(\alpha, z; T')$, then $a' \in \partial f(a)$ for some a in W, since

$$\mathcal{R}(\alpha, z; T') = \left\{x' \in T' : \langle x', z\rangle \geq \sup_{x' \in T'} \langle x', z\rangle - \alpha\right\}.$$

Since W is open, there exists $r > 0$ (sufficiently small) such that $b := a + rz \in W$. If $b' \in \partial f(b)$, therefore,

$$\langle b', a - b\rangle \leq f(a) - f(b) \quad and \quad \langle a', b - a\rangle \leq f(b) - f(a),$$

consequently,

$$\langle b', a - b\rangle + \langle a', b - a\rangle \leq 0. \tag{6}$$

As $a - b = -rz$ and $b - a = rz$ by replacing in (6)

$$0 \geq \langle b', -rz\rangle + \langle a', rz\rangle = r(\langle a', z\rangle - \langle b', z\rangle)$$

and how $r > 0$

$$\langle b', z\rangle \geq \langle a', z\rangle > \sup_{a' \in T'} \langle a', z\rangle - \alpha.$$

We have, thus, shown that $\partial f(b) \subseteq \mathcal{R}'(\alpha, z; T')$. Now, the set $\mathcal{R}(\alpha, z; T')$ is $\sigma(X', X)$-open in X' and as the subdifferential function is $\left(\| \|_X, \sigma(X', X)\right)$-continuous, there exists a $\delta > 0$ such that

$$\|x - b\|_X < \delta \quad implies \quad \partial f(x) \subseteq \mathcal{R}(\alpha, z; T'). \tag{7}$$

Suppose that $\|h\|_X \leq 1$. Then $\partial f(b + th), \partial f(b - th) \subseteq \mathcal{R}'(\alpha, z; T')$ if $0 < t < \delta$ since

$$\|(b + th) - b\|_X = \|th\|_X \leq t < \delta \quad and \quad \|(b - th) - b\|_X = \|th\|_X \leq t < \delta.$$

If $u \in \partial f(b)$, $v \in \partial f(b + th)$, $w \in \partial f(b - th)$ and $0 < t < \delta$, then

$$\begin{aligned} u(b + th) - u(b) = u(b + th) &\leq f(b + th) - f(b) \\ v(b) - u(b + th) = v(-th) &\leq f(b + th) - f(b) \\ w(b) - w(th) = w(b + th) &\leq f(b) - f(b - th). \end{aligned}$$

Now, as $0 < t < \delta$ and by the linearity of $u,\ v,\ w$ we have

$$\begin{aligned} u(h) &\leq \frac{f(b + th) - f(b)}{t} \leq v(h) \\ -u(h) &\leq \frac{f(b - th) - f(b)}{t} \leq -w(h). \end{aligned}$$

By (7) we know that $v,\ w \in \mathcal{R}'(\alpha, z; T')$ and that the diameter of this set is $< \varepsilon$, then we obtain

$$0 \;\leq\; \frac{f(b+th)+f(b-th)-2f(b)}{t} \;\leq\; v(h)-w(h) \;\leq\; \|v(h)-w(h)\|_{X'} <\; \varepsilon.$$

This last inequality is true for every $h \in X$ such that $\|h\|_X \leq 1$ and every $0 < t < \delta$. Therefore, $b \in D(f,\varepsilon)$, i.e., $W \cap D(f,\varepsilon) \neq \emptyset$. This means that $D(f,\varepsilon)$ is dense and in consequence

$$R := \bigcap_{n\in\mathbb{N}} A_n$$

(set of differentiability of f) is a G_δ-dense set. We have, thus, proved that X is an Asplund F-space.

5 The Class of Stegall $\mathfrak{S}$ on Topological Spaces

This section is the central part of the article, since it introduces an intermediate class between the Asplund F-spaces and $\mathcal{G}$-Asplund, called the class of Stegall $\widehat{\mathfrak{S}}$. This class establishes sufficient conditions for a Banach space to be a $\mathcal{G}$-Asplund.

Let us recall that a topological space (X,τ) is *completely regular*, if it is Hausdorff and for each closed set C and every point p that does not belong to C, there is a continuous function $f: X \to [0,1]$ such that $f|_C = 0$ and $f(p) = 1$. The Urysohn's lemma, every metric space is completely regular.

Theorem 5.1 *If X is a separable Banach space and $K \subset X'$ is bounded, then the topology $\sigma(X',X)|_K$ is metrizable (the topology of the dual is metrizable on the bounded sets).*

Theorem 5.2 *Let X be a Banach space. Then $\left(X',\sigma(X',X)\right)$ is a space completely regular.*

For the proof of previuos theorems see [(1)].

Definition 5.3 *(S-space) An S-space is a completely regular topological space X that satisfies the following condition:*

If Z is a Baire space and $f: Z \to X$ is a minimal usco function, then f is single-valued in a residual subset of Z. We will denote by $\mathfrak{S}$, the set of all the S-spaces.

Even though Marian J. Fabian in [(1)] proves the following theorems, some important details were left out. We present here complete proofs.

Theorem 5.4 *Every metric space (X,d) is an S-space.*

Proof. Let (X,d) be a metric space, Z be a Baire space and $f: Z \to X$ is a minimal usco function. For each $n \in \mathbb{N}$, we define the open set

$$\mathcal{U}_n = \cup\left\{V \subset Z : V \;\; open \;\; and \;\; diam\big(f(V)\big) < \tfrac{1}{n}\right\}. \tag{8}$$

We will show that $G = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$ is residual. Note that each one $\mathcal{U}_n$ is open and we will prove that each $\mathcal{U}_n$ is dense for all $n \in \mathbb{N}$. Let $x \in \mathcal{U}_n$ and $V_x \subset Z$ be an open neighborhood of x with

$$diam\big(f(V_x)\big) \ < \ \frac{1}{n}.$$

If $y \in f(x)$ and $B\left(y,\frac{1}{3n}\right)$, then $f(x) \cap B\left(y,\frac{1}{3n}\right) \neq \emptyset$ and by lemma 3.10.(ii), there is a nonempty open subset Ω of V_x such that $f(x') \subseteq B\left(y,\frac{1}{3n}\right)$ for each $x' \in \Omega$. So

$$diam\big(f(\Omega)\big) \ \le \ \frac{1}{3n}.$$

Then, $\Omega \subset \mathcal{U}_n$ and, therefore, $\emptyset \neq V_x \cap \mathcal{U}_n$, which proof the density of the $\mathcal{U}_n$. So $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a countable collection of open and dense subsets in Z and, therefore, the intersection G of the $\mathcal{U}_n$ is a residual subset. Finally, by vertue of theorem 3.14, the usco function f is single-valued in G and, consequently, $X \in \mathfrak{S}$.

Theorem 5.5 *Let X be a completely regular space. If $(X_n)_{n\in\mathbb{N}}$ is a sequence of closed subsets of X, with $X_n \in \mathfrak{S}$ for every $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} X_n$, then $X \in \mathfrak{S}$.*

Proof. Let Z be a Baire space and $f: Z \to X$ a minimal usco function. For each $n \in \mathbb{N}$, we define the set

$$Z_n := \{z \in Z : f(z) \cap X_n \neq \emptyset\}.$$

Since f is a upper semicontinuous function and by hypothesis each X_n is closed in X, then each Z_n is a closed set in Z by theorem 3.5. If $A_n = int(Z_n)$, then $A = \bigcup_{n\in\mathbb{N}} A_n$ is open and dense in Z, since Z is a space of Baire and $Z = \bigcup_{n\in\mathbb{N}} Z_n$ (see [(7)], p.63). It is clear that

$$A_n \neq \emptyset \quad for\ some \quad n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ be such that $A_n \neq \emptyset$, and let $f_n = f|_{A_n}$. We will show that f_n is a minimal usco function and $f_n(A_n) \subseteq X_n$.

1) In general, if $f: Z \to X$ is a usco function and $g: X \to Y$ (with Y a regular space) is a continuous single-valued function, by theorem 3.11.2, we have that $g \circ f$ is a usco function. From this it follows that if X is a subspace of Y, then $f|_X$ is a usco function since $f|_X = i_X \circ f$ and i_X is continuous.

2) We will show that f_n is minimal. For this we will use lemma 3.10 several times. Let $V \subseteq A_n$ open and $C \subseteq X$ closed such that

$$f_n(z) \cap C \neq \emptyset \quad for\ any \quad z \in V.$$

Then

$$f(z) \cap C \neq \emptyset \quad for\ any \quad z \in V$$

and as f is minimal, by lemma 3.10.(iii) then $f(z) \subseteq C$ for every $z \in V$. So

$$f_n(z) \subseteq C \quad for\ any \quad z \in V$$

and, hence, f_n is minimal (using lemma 3.10 again).

3) $f_n(A_n) \subseteq X_n$. By the definition Z_n, we have $f_n(z) \cap C \neq \emptyset$ for every $z \in A_n$ and by lemma 3.10.(iii), we obtain that $f_n(A_n) \subseteq X_n$ since X_n is closed by hypothesis.

Let

$$R := \{z \in Z : f(z)\ \ is\ a\ singleton\ for\ all\ \ z\}$$

and $R_n := R \cap A_n$.

We note that R_n is the set of $z \in A_n$ such that $f_n(z)$ consists of a single element. By hipothesis $X_n \in \mathfrak{S}$, so A_n is a Baire space because it is an open subset of a Baire space and $f_n: A_n \to X_n$ is a minimal usco function. Then we conclude that R_n is a residual set in A_n for each $n \in \mathbb{N}$. Now we can show that R is residual in Z. We have shown that there exists a countable collection $\{A_n\}_{n\in\mathbb{N}}$ of open sets of Z such that $R \cap A_n$ is residual in A_n and $A = \bigcup_{n\in\mathbb{N}} A_n$ is dense in Z. This implies that R is residual as we shall proof continuation. We define

$$W_1 = A_1 \ \ y \ \ W_n = A_n - adh\left(\bigcup_{k=1}^{n-1} A_k\right) \ \ if \ \ n \geq 2.$$

Then $\{W_n\}_{n\in\mathbb{N}}$ is an open and disjoint collection of open sets and $W_n \subseteq A_n$ for each $n \in \mathbb{N}$. Let us show that $W = \bigcup_{n\in\mathbb{N}} W_n$ is dense in Z. Let $z \in Z$ and V be a open neighborhood of z contained in Z. As $V \cap A \neq \emptyset$, then $V \cap A_n \neq \emptyset$ for some $n \in \mathbb{N}$. Let p be the first ordinal such that $V \cap A_p \neq \emptyset$. Then $V \cap W_p \neq \emptyset$ and, therefore, $V \cap W = V \cap \bigcup_{n\in\mathbb{N}} W_n \neq \emptyset$ which shows that W is dense at Z.

Since $R \cap A_n$ is residual in A_n, there exist a countable collection $\{T_{kn} : k \in \mathbb{N}\}$ of open and dense subsets of A_n such that $R \cap A_n \supseteq \bigcap_{k\in\mathbb{N}} T_{kn}$. Now, since $W_n \subseteq A_n$, then

$$R \cap W_n \supseteq W_n \cap (R \cap A_n) \supseteq \bigcap_{k\in\mathbb{N}} (W_n \cap T_{kn}). \qquad (9)$$

The set $W_n \cap T_{kn}$ is open and dense in W_n. In fact, let $z \in W_n$ and V be a open neighborhood of z contained in W_n. Then V is an open neighborhood of z contained in A_n. Since T_{kn} is dense in A_n, then $V \cap T_{kn} \neq \emptyset$ and, therefore, $V \cap (W_n \cap T_{kn}) \neq \emptyset$. Since W_n is a Baire space, then

$$W_n \cap \left(\bigcap_{k\in\mathbb{N}} T_{kn}\right) = \bigcap_{k\in\mathbb{N}} (W_n \cap T_{kn}) \neq \emptyset.$$

This is the final part of the proof. Let $T_k = \bigcup_{n\in\mathbb{N}} (T_{kn} \cap W_n)$. Then we will see that T_k is open and dense in Z. To prove this statement, let V be an open and nonempty subset of Z. Since W is dense in Z, then $W \cap V \neq \emptyset$ and, therefore, $V \cap W_n \neq \emptyset$ for some $n \in \mathbb{N}$ (since W is defined as the union of W_n). As $W_n \cap T_{kn}$ is open and dense in

W_n, it follows that

$$(V \cap W_n) \cap (T_{kn} \cap W_n) = V \cap (T_{kn} \cap W_n) \neq \emptyset.$$

We have, thus, shown that $V \cap T_k \neq \emptyset$ for any open and nonempty of Z, i.e., T_k is open and dense in Z. As Z is a Baire space, then $T := \bigcap_{k\in\mathbb{N}} T_k$ is dense in Z. Now we remark that

$$T = \bigcap_{k\in\mathbb{N}} T_k = \bigcap_{k\in\mathbb{N}} \left(\bigcup_{n\in\mathbb{N}} (T_{kn} \cap W_n) \right).$$

As

$$W_n \cap T_k = \bigcup_{m\in\mathbb{N}} \big(W_n \cap (T_{km} \cap W_m)\big) = T_{kn} \cap W_n$$

since $W_n \cap W_m = \emptyset$ if $n \neq m$, then

$$\bigcap_{k\in\mathbb{N}} T_k = \bigcap_{k\in\mathbb{N}} \left(\bigcup_{n\in\mathbb{N}} (W_n \cap T_{kn}) \right) = \bigcap_{k\in\mathbb{N}} \left(T_{kn} \cap \left(\bigcup_{n\in\mathbb{N}} W_n \right) \right)$$
$$= \bigcup_{n\in\mathbb{N}} \left(\bigcap_{k\in\mathbb{N}} (W_n \cap T_{kn}) \right).$$

Using (9), we obtain that

$$\bigcap_{k\in\mathbb{N}} T_k \subseteq \bigcup_{n\in\mathbb{N}} (R \cap W_n) \subseteq R.$$

In other words, we have shown that R is residual, since each T_k is open and dense, accordingly $X = \bigcup_{n\in\mathbb{N}} X_n \in \mathfrak{S}$.

5.1 The Class of Stegall $\widehat{\mathfrak{S}}$ of Banach Spaces

The class $\widehat{\mathfrak{S}}$ of *Stegall spaces* consists of all of the Banach spaces whose dual with the weak- * topology are topological $\mathfrak{S}$-spaces. In other words, $X \in \widehat{\mathfrak{S}}$, if X is a Banach space and for all Baire space Z, we have that any minimal usco function T of Z in X' (with weak- * topologie) is single-valued in a residual subset.

Next, we give a detailed proof of the Stegall's theorem whose proof can be found in [(1)] citing numerous articles. We show that the usco multivalued functions, particulary the subdifferentials of convex functions, play an important role in this proof.

Theorem 5.1.1 *The Stegall class lies between the Asplund F-spaces and the $\mathcal{G}$-Asplund. In other words,*

$$\mathfrak{U}_F \subseteq \widehat{\mathfrak{S}} \subseteq \mathfrak{U}_G.$$

Proof. Let us first show that $\mathfrak{U}_F \subseteq \widehat{\mathfrak{S}}$. Let X be a Banach space and X' be its dual. Recall that a bounded subset M of X' is weak *-dentable, if it admits weak- $^*(\alpha, x)$-slice of arbitrarily small diameter. The weak- $^*(\alpha, x)$-slice of M is the set

$$\mathcal{R}(\alpha, x; M) = \{x' \in M: \langle x, x'\rangle > \sup\langle x, M\rangle - \alpha\}$$

A weak- $^*(\alpha, x)$-slice of M is a set $\sigma(X', X)|_M$ open due to the continuity of the function $x' \mapsto \langle x, x'\rangle$ (i.e., the continuity of the support function).

Suppose X is an Asplund F-space and let $M = B_{X'} = \{x' \in X': \|x'\|_{X'} \leq 1\}$. Let Z be a Baire space and $T: Z \to (M, \tau)$, where $\tau = \sigma(X', X)|_M$ (restriction of $\sigma(X', X)$ to M) and let T be a minimal usco function. We will show that T is single-valued in a residual subset. This would show that M is an S-space and since M is $\sigma(X', X)$-closed and

$$X' = \bigcup_{n\in\mathbb{N}} nM,$$

(since X' is fitted with the topology $\tau = \sigma(X', X)|_{B_{X'}}$ and $B_{X'} \in \tau(0)$) we conclude that X' is an S-space by theorem 5.5, that is, $X \in \widehat{\mathfrak{S}}$.

Our aim now is to show that T is single-valued in a residual subset. For this, let us consider the set defined in (8).

We claim that $\mathcal{U}_n$ is an open set in (M, τ), where $\tau = \sigma(X', X)|_M$. In fact, if $w \in V$, then V is an open neighborhood of w.

Now, we claim that $\mathcal{U}_n$ is dense in Z. To prove this statement, let $\mathcal{U}$ be an open and nonempty subset of Z. Recall that X is an Asplund space, so from theorem 4.1.4 Theorem's Namioka Phelps, X' is weak- *-dentable. Now, since $T(\mathcal{U}) \subseteq M$, then given $\varepsilon > 0$ there exists a weak- $^*(\alpha, x)$-slice $\mathcal{R}(\alpha, x; T(\mathcal{U}))$ of arbitrarily small diameter, let say $< \varepsilon$ (i.e., $diam[\mathcal{R}(\alpha, x; T(\mathcal{U}))] < \varepsilon$). If

$$W := \{x' \in M: \langle x, x'\rangle > \sup\langle x, M\rangle - \alpha\},$$

then W is a subset τ-open of M and $T(\mathcal{U}) \cap W = \mathcal{R}(\alpha, x; T(\mathcal{U})) \neq \emptyset$ since

$$\mathcal{R}(\alpha, x; T(\mathcal{U})) = \{x' \in T(\mathcal{U}): \langle x, x'\rangle > \sup\langle x, T(\mathcal{U})\rangle - \alpha\}.$$

By lemma 3.10.(ii), there exists an open nonempty subset $S \subseteq \mathcal{U}$ such that $T(S) \subseteq W$ and, therefore, $T(S) \subseteq T(\mathcal{U})$ and $T(S) \subseteq W \cap T(\mathcal{U}) = \mathcal{R}(\alpha, x; T(\mathcal{U}))$. So that,

$$diam(T(S)) < \varepsilon \quad and \quad S \subseteq \mathcal{U}_n \cap T,$$

which shows that $\mathcal{U}_n$ is dense in Z. From previous claims, we have that $\{\mathcal{U}_n: n \in \mathbb{N}\}$ is a countable collection of open and dense subsets in Z. We remark that

$$\bigcap_{n\in\mathbb{N}} \mathcal{U}_n \subseteq \{z \in Z: T(z) \ \ is\ a\ singleton\ for\ all\ \ z\} := \mathcal{D}.$$

Indeed, if $z \in \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$, then

$$diam\big(T(z)\big) \ < \ \frac{1}{n} \quad for\ any \quad n \in \mathbb{N},$$

that is, $T(z)$ is a singleton for all z from the dense residual subset $\mathcal{D}$ of Z. We have, thus, shown that T is single-valued in a residual subset of Z, and so $X \in \widehat{\mathfrak{S}}$.

Now, we will establish that $\widehat{\mathfrak{S}} \subseteq \mathfrak{U}_G$. Let X be a Banach space in Stegall's class $\widehat{\mathfrak{S}}$. That is, $\big(X', \sigma(X', X)\big)$ is an S-space ($\big(X', \sigma(X', X)\big) \in \mathfrak{S}$). That is, any minimal usco function of a Baire space Z in X' is single-valued in a residual subset. We must show that X is a $\mathcal{G}$-Asplund space. Let $\psi: \Omega \to \mathbb{R}$ be a continuous convex function, where Ω is a convex open subset of X and show that this function is G-differentiable in a residual subset of Ω. The subdifferential of ψ is the multivalued function $\partial\psi: \Omega \to X'$ defined as:

$$x \mapsto \partial\psi(x) = \{\psi(x+h) - \psi(x) \ \ \geq \ \ u(h) \ \ for\ any\ \ h \ \ such\ that \ \ x+h \in \Omega\}.$$

By theorem 3.1.7, the subdifferential $\partial\psi$ is a usco function considering in Ω the norm topology and in X' the weak-* topology $\sigma(X', X)$. By theorem 3.7, there exists a minimal usco function $T: \Omega \to X'$ such that $T \preccurlyeq \partial\psi$. Since Ω is a Baire space, $\big(X', \sigma(X', X)\big)$ is completely regular and $T: \Omega \to X'$ is a minimal usco function, then T is single-valued on a residual subset $\mathcal{D}$ of Ω. It remains to verify that ψ is G-differentiable in $\mathcal{D}$. Let $x \in \mathcal{D}$ (arbitrary but fixed) and $h \in X$. Then $T(x) = \{u\}$. As $T(x) \subseteq \partial\psi(x)$, then $u \in \partial\psi(x)$ and, therefore, for a $t > 0$ small enough we have to

$$\psi(x+th) - \psi(x) \ \ \geq \ \ u(th) \quad with \quad x+th \in \Omega. \tag{10}$$

Let $v_t \in T(x+th)$. Then $v_t \in \psi(x+th)$ and, therefore, by changing the variable $z = x+th$, we have that

$$\psi\big(z + (-th)\big) - \psi(z) \ \ \geq \ \ v_t(-th)$$
$$\psi(x) - \psi(x+th) \ \ \geq \ \ v_t(-th),$$

which implies that

$$\psi(x+th) - \psi(x) \ \ \leq \ \ v_t(th). \tag{11}$$

Since $t > 0$, from (10) and (11), we obtain that

$$u(h) \ \ \leq \ \ \frac{\psi(x+th) - \psi(x)}{t} \ \ \leq \ \ v_t(h)$$

and, therefore,

$$0 \ \ \leq \ \ \tfrac{\psi(x+th)-\psi(x)}{t} - u(h) \ \ \leq \ \ v_t(h) - u(h). \tag{12}$$

Now, let $\varepsilon > 0$ be such that

$$W = \{\xi \in X': |\xi(h) - u(h)| < \varepsilon\}$$

is a $\sigma(X',X)$-open neighborhood $u \in X'$. Using the upper semicontinuity of T in $x \in \Omega$, there exists an open neighborhood $V \subseteq \Omega$ of x such that

$$T(z) \subseteq W \quad for\ any \quad z \in V.$$

There exists a $\delta > 0$ such that $x + th \in V$ if $0 < t < \delta$. So that,

$$T(x+th) \subseteq W \quad if \quad 0 < t < \delta$$

and, therefore,

$$v_t(h) - u(h) \ < \ \varepsilon \quad if \quad 0 < t < \delta. \tag{13}$$

In virtue of (12) and (13), we conclude that

$$\lim_{t\to 0^+} \frac{\psi(x+th)-\psi(x)}{t} = u(h). \tag{14}$$

Since $u \in X'$, so

$$\lim_{t\to 0} \frac{\psi(x+th)-\psi(x)}{t} = u(h) \ and \ \psi \ is \ G - differentiable\ in \ x.$$

We have, thus, shown that ψ is G-differentiable in a G_δ-dense subset $\mathcal{D}$ and hence $X \in \mathfrak{U}_G$.

Proposition 5.1.2 *Every separable Banach space belongs to the Stegall class.*
Proof. Suppose X is a separable Banach space. Let

$$B_{E'} = \{x' \in X' : \|x\|_{X'} \leq 1\} \quad and \quad \tau = \sigma(X',X)|_{B_{X'}}.$$

By the Banach-Alaoglu theorem, the closed unitary ball B' de X' is $\sigma(X',X)$-compact, then by theorem 5.1 we have that $(B_{X'},\tau)$ is metrizable and $(B_{X'},\tau) \in \mathfrak{S}$. As

$$X' = \bigcup_{n\in\mathbb{N}} nB_{X'} \quad and \quad B_{X'} \ is \quad \sigma(X',X) - closed,$$

then $\left(X', \sigma(X',X)\right) \in \mathfrak{S}$ by theorem 5.5 and therefore $X \in \widehat{\mathfrak{S}}$. We have, thus, shown that X is a Stegall space.

Remarks 5.1.3

1) In general $\mathfrak{U}_F \subset \widehat{\mathfrak{S}}$ (strict inclusion). Indeed, let us consider $X = \ell^1(\mathbb{R})$. It has been showed that the $\|\cdot\|_1$ is not F-differentiable at any point, but by theorem 5.1.2 we deduce that $\ell^1(\mathbb{R})$ is a Stegall space because it is separable. Consequently, $\ell^1(\mathbb{R}) \in \mathfrak{S}$, but $\ell^1(\mathbb{R}) \notin \mathfrak{U}_F$.

2) In general $\widehat{\mathfrak{S}} \subset \mathfrak{U}_G$(strict inclusion). To prove this result, we should go beyond the separable Banach spaces. Indeed, it was shown in theorem 5.1.2 that a

separable Banach space is a Stegall space. Furthermore, Mazur's theorem assures us that the separable Banach spaces are also $\mathcal{G}$-Asplund spaces (see [8]). Kalenda in [8] showed that there are $\mathcal{G}$-Asplund spaces whose dual with the weak weak-* topology do not belong to class $\widetilde{\mathfrak{S}}$. That is, there exist $\mathcal{G}$-Asplund spaces which are not in the $\widetilde{\mathfrak{S}}$ class.

3) There exist Banach spaces that are not $\mathcal{G}$-Asplund spaces, for example, the nonseparable Banach space $X = \ell^{\infty}(\mathbb{R})$ is not a $\mathcal{G}$-Asplund space.

5.2 Example of Usco Function

The Supremum Mapping

As it is well known, the problem of the differentiability of convex functions is well addressed by Fabian in [1]. Let X be a compact and Hausdorff topological space, and $\mathcal{C}(X)$ be the Banach space of the continuous functions of X in $\mathbb{R}$ with the norm $\|f\|_{\infty} := \sup_{t\in X}|f(t)|$. We define the supreme function on $\mathcal{C}(X)$ as

$$\varphi: \mathcal{C}(X) \to \mathbb{R}$$

$$f \mapsto \varphi(f) := \sup_{t\in X}|f(t)|.$$

This function is sublineal and, therefore, convex. In addition, we define

$$\psi: \mathcal{C}(X) \to X$$
$$f \mapsto \psi(f) := \{t \in X: \varphi(f) = f(t)\}.$$

This is a multivalued function that assigns to each function f the set of points $t \in X$ in which the function attains the supremum. We will call ψ the *supremum mapping*. We remark that the function $f \mapsto \psi(f)$ is a usco function. Indeed,

I. It is clear that $\psi(f)$ is compact because

$$\{t \in X: \varphi(f) = \beta = f(t)\}$$

is closed in X. As X is compact,

$$\varphi(f) \neq \emptyset \quad for\ everything\ f \in \mathcal{C}(X)$$

and, therefore, $\psi(f) \neq \emptyset$ for all $f \in \mathcal{C}(X)$.

II. We will show that ψ is upper semicontinuous. We argue by contradiction assuming that for some point $g \in \mathcal{C}(X)$ there is an open neighborhood $\psi(g) \subseteq W$ such that on every open ball $B(g,\varepsilon)$ there exists a h such that $\psi(h)$ is not contained in W. That is, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that $\|g_n - g\| \to 0$ and $\psi(g_n) \cap W^c \neq \emptyset$. Let $t_n \in \psi(g_n) \cap W^c$ and we remark that

$$\begin{aligned}\varphi(g) &\leq \varphi(g-g_n) - \varphi(g_n) = \varphi(g-g_n) + (g_n - g)(t_n) + g(t_n)\\ &= 2\|g-g_n\|_{\infty} + g(t_n).\end{aligned}$$

As W^c is compact, the sequence $(t_n)_{n\in\mathbb{N}}$ has limit value $t \in W^c$. Since g is a continuous function, $g(t)$ is an adhesion value of the sequence $\left(g(t_n)\right)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Therefore, there is a subsequence $\left(t_{k_n}\right)_{n\in\mathbb{N}}$ such that $g\left(t_{k_n}\right) \to g(t)$ and how $\|g_n - g\| \to 0$, then $\varphi(g) \le g(t)$. In addition, $g(t) \le \varphi(g)$, and, therefore, $\varphi(g) = g(t)$, i.e., $t \in \psi(g) \subseteq W$ which contradicts that $t \in W^c$. We have shown that ψ is upper semicontinuous.

In summary, ψ is a usco function. We emphasize that this example is closely related to the differentiability of the supremum norm in $\mathcal{C}(X)$.

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