Facultad de Ciencias Naturales y Exactas
Universidad del Valle

A NOTE ON THE STABILITY AND INSTABILITY OF TRAVELLING WAVE OF KORTEWEG-DE VRIES TYPE: THE PERIODIC CASE

José R. Quintero
Universidad del Valle

Received: July 1, 2010 Accepted: October 6, 2010

Pág. 57-72

Abstract

In this paper we adapt the work of M. Grillakis, J. Shatah, and W. Strauss, or J. Bona, P. Souganidis and W. Strauss to the periodic case in spaces having the mean zero property in order to establish the orbital stability/instability of periodic travelling wave solutions of a generalized Korteweg-de Vries type equation.

Keywords: KdV type models; periodic travelling waves; orbital stability/instability.
MSC (2000): 35Q53, 35B35, 76B30.

1 Introduction

In this work we analyze the stability and instability of periodic travelling wave solutions $u(x,t) = \varphi(x-ct)$ ($\zeta = x - ct$ periodic) of a generalized Korteweg-de Vries class of evolution equations of the form

$$u_t + u_x - Mu_x + (u^p)_x = 0, \tag{1}$$

where $u(x,t)$ is a real valued function, $p > 1$ is an integer, and M is a constant coefficient pseudo-differential operator of order $\mu > 1$, having the form of a multiplier operator of the following type

$$\widehat{Mu}(k) = |k|^{\mu}\hat{u}(k), \quad k \in \mathbb{Z},$$

where $\hat{u}(k)$ denotes the k-Fourier coefficient of u.

It is well known, these models describe the unidirectional propagation of weakly nonlinear, dispersive, long waves with small amplitude. For $M = -\partial_x^2$ and $p = 2$, the equation (1) corresponds to the classical model derived in 1895 by Korteweg and de Vries for surface water waves in a canal see [3]

$$u_t + u_x + u_{xxx} + (u^2)_x = 0. \tag{2}$$

For $M = -\partial_x^2$ and $p = 3$, equation (1) is known as the modified Korteweg and de Vries equation

$$u_t + u_x + u_{xxx} + (u^3)_x = 0. \tag{3}$$

The well known Benjamin-Ono equation is associated with $\mu = 1$ and $p = 2$. In this case, $M = \mathcal{H}\partial_x$, where $\mathcal{H}$ denotes the periodic Hilbert transform, having Fourier coefficients of the form $\widehat{\mathcal{H}f}(k) = -isign(k)\hat{f}(k)$, and equation (1) takes the form

$$u_t + u_x - \mathcal{H}u_{xx} + (u^2)_x = 0. \tag{4}$$

Regarding the KdV equation (2) in the periodic case, Angulo, Bona and Scialom in [4] showed the existence of a branch of *cnoidal* ($c \to \varphi_c$) solutions with a fixed period L and having mean zero in $[0.L]$. They also established the nonlinear stability of the orbit $\{\varphi_c(\cdot + y) : y \in \mathbb{R}\}$ in the space $H^1_{per}([0,L])$ and also in the closed subspace of $H^1_{per}([0,L])$ defined as

$$\mathcal{W}^1_L = \left\{ f \in H^l_{per}([0,L]) : \int_0^L f(x)\,dx = 0 \right\}.$$

For Instability of $2L$-periodic *cnoidal* waves, N. Bottman and B. Deconinck showed that the waves are spectrally stable analytically and numerically see [5]. Also B. Deconinck and T. Kapitula proved that the periodic travelling waves for the KdV model are orbitally stable subject to perturbations which are nL-periodic for any given $n \in \mathbb{N}$ see [6]. We must point out that M. Johnson established an orbital stability result in the same fashion for the case $M = -\partial_x^2$, as the one presented here see [7]. In the later work, M. Johnson followed some of the ideas used in the case of the orbital stability of waves to the nonlinear Schrödinger equation obtained by T. Gallay and C. Hărăgus in[8].

In a recent paper, Angulo and Natali in [9] obtained H^1_{per}-stability results for the KdV model (2) ($p = 2$), the generalized KdV model (3) ($p = 3$), and the BO model (4) ($p = 2$). They make use of the theory of totally positive operators, the Poisson summation theorem and the theory of Jacobian elliptic functions. We also have to mention that Gardner in [10] obtained an instability result for periodic travelling waves of equation (1) when $p > 5$. In that paper, it is assumed the existence of a family of large wavelenght periodic waves U^σ such that the period $2T_\sigma$ tends to infinity as σ tends to zero. Gardner proved instability of the travelling wave U^σ, for $p > 5$ and for $\sigma > 0$, but small enough.

In order to discuss the stability/instability issue, we note that the generalized model (1) has the Hamiltonian form

$$u_t = \mathcal{J}E'(u)$$

where $\mathcal{J} = -\partial_x$ and E is the functional defined in $H^{\frac{\mu}{2}}_{per}([0,L])$ or $\mathcal{W}^{\frac{\mu}{2}}_L$ as

$$E(u) = \int_0^L \left(\frac{1}{2}uMu - \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1} \right) dx,$$

with $\mathcal{W}_L^s$ denoting the closed subspace of $H^s_{per}([0,L])$ given by

$$\mathcal{W}_L^s = \left\{ f \in H^s_{per}([0,L]) : \int_0^L f(x)\,dx = 0 \right\}, \quad s \in \mathbb{R}.$$

We note that $f \in \mathcal{W}_L^s$ if and only if $f(x) = \sum_{-\infty}^{\infty} \hat{f}_n e^{\frac{2\pi i n x}{L}}$ with $\hat{f}_0 = 0$, and

$$\sum_{-\infty}^{\infty} \left(1 + \frac{4\pi^2 |n|^2}{L^2}\right)^s |f_n|^2 < \infty.$$

As it is well known, E and the following functionals are time invariants of the motion generated by equation (1)

$$V(u) = \int_0^L u^2\,dx,$$
$$I(u) = \int_0^L u\,dx.$$

We will see that under certain conditions periodic travelling waves φ_c, for $c > 1$ are stable if and only if

$$d(c) = \mathcal{F}(\varphi_c) = E(\varphi_c) + cV(\varphi_c)$$

is a convex function of c.

If we consider the KdV type models (2), (3) or (4), we are almost in the exact setting of Grillakis *et al.* general results in [1] in the sense that the operator $\partial_x : \mathcal{W}_L^s \to \mathcal{W}_L^{s-1}$ is a bijection. It is very important to note that in those particular cases, the eigenfunction $\chi_c \in H^1_{per}([0,L])$, associated with the unique negative simple eigenvalue of $\mathcal{F}''(\varphi_c)$ does not satisfy the mean zero property in $[0,L]$, since due to an oscillation argument the eigenfunction χ_c can be taken either positive or negative. In other words, the result of stability/instability in either Grillakis *et al.* in [1] or by Bona *et al.* in [2] does not apply directly in the case of having the mean zero property.

Regarding instability, we observe that the result for fixed c obtained by Grillakis *et al.* in [1] (or by Bona *et al.* in [2]) relies on three basic facts:

1. the existence of a curve parametrized by the wave speed

   $$\omega \to \psi_\omega = \varphi_w + s(\omega)\chi_c \in H^{1+\frac{\mu}{2}}(\mathbb{R})$$

   (χ_c is the eigenfunction associated with the unique negative eigenvalue of $\mathcal{F}''(\varphi_c)$),

2. the existence of an element $y = \partial_\omega \psi_\omega|_{\{c=\omega\}} \in H^{\frac{\mu}{2}}(\mathbb{R})$ such that $\langle \mathcal{F}''(\varphi_c)y, y\rangle < 0$, and

3. the existence of a Liapunov type functional B defined in a tube in $H^{\frac{\mu}{2}}(\mathbb{R})$.

Even though there are some recent results related with the stability/instability of periodic travelling wave solutions for the KdV equation, which could be more general or whose proofs use more sophisticated techniques, we consider a classical strategy to analyze stability and instability in the sense that we just adapt the work of either M. Grillakis, J. Shatah, and W. Strauss [1], or J. Bona, P. Souganidis and W. Strauss [2] to the periodic case using a space with the mean zero property. In particular,

1. we will see how to build a curve $\omega \to \psi_\omega$ on the mean zero space $\mathcal{W}_L^{1+\frac{\mu}{2}}$,
2. we will show the existence of an element $y \in W_L^{\frac{\mu}{2}}$ satisfying that $\langle \mathcal{F}''(\varphi_c)y, y\rangle < 0$, and
3. we will establish the existence of a Liapunov type functional B defined in a tube in $\mathcal{W}_L^{\frac{\mu}{2}}$.

In other words, we will show that the classical approach used by Grillakis *et al.* in [1] (or by Bona *et al.* in [2]) to establish stability/instability of solitons still holds in spaces with the mean zero property to analyze stability/instability of L-periodic travelling waves.

This paper is organized as follows. In Section 2, we consider the analysis of periodic travelling waves solutions for equation (1). We will see that these solutions are critical points of the functional E subject to the constrain V constant. In Section 3, we adapt to the periodic case the work by Grillakis *et.al* [1] or Bona *et al.* [2]. One important fact in case $d''(c) < 0$ is that these critical points are saddle points but not local minimum, in contrast with the case $d''(c) > 0$, where these critical points are in fact local minimum. We also discuss the instability and the stability issue. In particular, we construct a Lyapunov type functional as done for soliton in ([1, 2]). Then the instability and stability follow exactly as in ([1, 2]). For the sake of completeness we include some of the proofs. In this work, we do not address the well posedness of the Cauchy problem associated to the generalized KdV equation (1) due to the huge number of references related with the generalized KdV model (see [12], [13], [14], [15], [16]).

2 Periodic travelling waves of the generalized KdV type model

Let assume that $c > 1$ and that $u(x,t) = \varphi_c(x-ct)$ is a solution of the evolution equation (1) with φ being a periodic function with the mean zero property in $[0, L]$. Then we have that φ_c satisfies the equation

$$-c\varphi_x - \partial_x M\varphi + \partial_x\varphi + \partial_x\varphi^p = 0$$

integrating this equation and using the periodicity of φ_c, then we find that φ_c also satisfies the equation

$$M\varphi + c\varphi - \varphi - \varphi^p = A. \tag{5}$$

We observe for p even that A must be different from zero due to the assumption on the mean property in $[0, L]$. Now we note that the previous equation has the form

$$E'(\varphi_c) + cV'(\varphi_c) = A$$

since

$$E'(u) = Mu - u - u^p, \quad \text{and} \quad V'(u) = u.$$

Moreover, φ_c is a critical point for $E(u) + cV(u)$ in a space having the mean zero property in $[0, L]$. In fact, let v have the mean zero property in $[0, L]$, then

$$\langle E'(\varphi_c) + cV'(\varphi_c), v\rangle = \langle A, v\rangle = A \int_0^L v(x)\,dx = 0. \tag{6}$$

On the other hand,

$$E''(u) = M - 1 - pu^{p-1}$$

and the linearized operator $\mathcal{L}_c$ of the operator $\mathcal{F}'$ in φ_c, around φ_c, is giving by

$$\mathcal{L}_c = \mathcal{F}''(\varphi_c) = (M + c - 1) - p\varphi_c^{p-1}.$$

The first observation is that $\lambda = 0$ is an eigenvalue of $\mathcal{L}_c$, since $\mathcal{L}_c(\partial_x\varphi_c) = 0$ and $\partial_x\varphi_c$ trivially has the mean zero property in $[0, L]$. In some case, it is possible to establish that $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}_c$ ([9], [18]). Before we go further, we observe that $\mathcal{L}_c$ is a linear, closed, unbounded, self adjoint operator defined on a dense subspace of $L^2_{per}([0, L])$ denoted with $D(\mathcal{L}_c)$ and range $L^2_{per}([0, L])$. Now, if we consider the periodic eigenvalue problem associated with $\mathcal{L}_c$

$$\begin{cases} \mathcal{L}_c v &= \lambda v \\ v(0) &= v(L), \quad v'(0) = v'(L) \end{cases} \tag{7}$$

we have from the theory of compact symmetric operators that the spectrum of $\mathcal{L}_c$ has an enumerable (infinite) set of eigenvalues $(\lambda_k)_{k=0}^{\infty}$ such that $\lambda_k \to \infty$ as $k \to \infty$. More exactly,

Proposition 2.1 *The operator $\mathcal{L}_c$ is a closed, unbounded, self-adjoint operator on $L^2_{per}([0, L])$ whose spectrum consists in an enumerable (infinite) set of eigenvalues $(\lambda_k)_{k=0}^{\infty}$ satisfying*

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots, \quad \lambda_k \to \infty, \quad \textit{as} \;\; k \to \infty.$$

Moreover, $\mathcal{L}_c$ has $\lambda = 0$ as an eigenvalue with eigenfunction $\partial_x\varphi_c$.

Proof. It is straightforward to see that $\mathcal{L}_c$ is a closed, unbounded, self-adjoint operator on $L^2_{per}([0, L])$. As we discuss above, $\lambda = 0$ is an eigenvalue since $\mathcal{L}_c(\partial_x\varphi_c) = 0$ and $\partial_x\varphi_c$ has the mean zero property in $[0, L]$.
Now consider the operator defined as $K_c = M + c - 1$. We observe that the representation of K_c is given by the Fourier symbol $\widehat{K}_c(n) = |n|^{\mu} + c - 1$, for $n \in \mathbb{Z}$. Moreover, since $c > 1$ the operator K_c is invertible with Fourier symbol $\frac{1}{|n|^{\mu}+c-1} \in l(\mathbb{Z})$, since $\mu > 1$. As a consequence of this, there is a unique operator $K_c^{-1} = N_c \in \mathcal{L}_b(L^2_{per}([0, L]))$ with Fourier symbol $\widehat{N}_c(n) = \frac{1}{|n|^{\mu}+c-1}$. Moreover, the representation of this operator is given by

$$N_c u(x) = \sum_{n\in\mathbb{Z}} \left(\frac{\hat{u}_n}{|n|^\mu + c - 1}\right) e^{\frac{2\pi i n x}{L}}$$

Clearly, the operator N_c is self-adjoint and compact on $L^2_{per}([0,L])$ for $c > 1$ and $\mu > 1$.
Now we observe that for some $\rho > 0$, the operator $M_\rho = \mathcal{L}_c + \rho$ is positive. In fact, let $u \in D(\mathcal{L}_c)$. Then using Parseval Theorem,

$$\begin{aligned}\langle \mathcal{L}_c u, u\rangle &\geq \frac{1}{L}\sum_{n\in\mathbb{Z}} \left((|n|^\mu + c - 1)|\widehat{u}(n)|^2\right)\, d\zeta - C(\varphi)\|u\|_0^2 \\ &\geq -\rho_1\|u\|_0^2, \quad \rho_1 = C(\varphi) - (c-1).\end{aligned}$$

Then for any $\rho \geq \rho_1$, the operator $M_\rho = \mathcal{L}_c + \rho$ is a positive and self-adjoint, and so is the operator M_ρ^{-1}. On the other hand,

$$\begin{aligned}M_\rho = \mathcal{L}_c + \rho &= M + c - 1 - p\varphi^{p-1} + \rho \\ &= M + \nu + \rho - 1 - \left(\nu - c + p\varphi^{p-1}\right) \\ &= K_{\nu+\rho} - P,\end{aligned}$$

where $P = \nu - c + p\varphi^p \in \mathcal{L}_b\left(L^2_{Per}\right)$ with $\nu > 0$ such that $\nu - c + p\varphi^p > 0$. As a consequence of this, for $\nu + \rho > 1$, we have that $K_{\nu+\rho}$ is invertible with inverse $N_{\nu+\rho}$. Then we conclude that

$$N_{\nu+\rho} \circ M_\rho = I_d - Z_{(\nu,\rho)}, \quad Z_{(\nu,\rho)} = N_{\nu+\rho} \circ P. \tag{8}$$

Since $N_{\nu+\rho}$ is a compact operator for $\nu + \rho > 1$ and $P \in \mathcal{L}_b\left(L^2_{Per}\right)$, then we conclude that $Z_{(\nu,\rho)}$ is compact. On the other hand, a direct computation shows that

$$\|Z_{(\nu,\rho)}\|_{\mathcal{L}_b(L^2_{Per})} \leq \sup_{n\in\mathbb{Z}} \left(\frac{1}{|n|^\mu + \nu + \rho - 1}\right)(\nu + c + C_1(\|\varphi\|_\infty))$$

then, taking ρ sufficiently large such that the operator $\|Z_{(\nu,\rho)}\|_{\mathcal{L}_b(L^2_{Per})} < 1$, we have that $(I - Z_{(\nu,\rho)})^{-1} \in \mathcal{L}_b\left(L^2_{Per}\right)$. Moreover, from the equality (8) we conclude that M_ρ^{-1} is compact since $N_{\nu+\rho}$ is compact, $(I - Z_{(\nu,\rho)})^{-1} \in \mathcal{L}_b\left(L^2_{Per}\right)$, and that $M_\rho^{-1} = (I_d - Z_{(\nu,\rho)})^{-1} N_{\nu+\rho}$. Thus from the spectral theorem for compact self-adjoint operator, we infer that there is an orthonormal basis $(\varphi_k)_k \subset L^2_{Per}$ which are the eigenfunctions of M_ρ^{-1} with non-zero eigenvalues $(\sigma_k)_k$ such that

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdot\cdot\cdot > 0, \quad \sigma_k \to 0, \quad k \to \infty.$$

From this, we conclude that

$$\mathcal{L}_c\varphi_k = \lambda_k\varphi_k,$$

where $\lambda_k = \left(\frac{1}{\sigma_k} - \rho\right)\varphi_k$. Then we have shown the existence of sequence of eigenvalues $(\lambda_k)_k$ for $\mathcal{L}_c$ such that

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_2 \leq \cdot\cdot\cdot, \quad \lambda_k \to \infty, \quad k \to \infty.$$ □

It is also important to see that depending on the form of the operator M, it is possible to show the existence of a branch of solutions φ_c (parametrized by the wave speed c), having a fixed minimal period L. For instance, in case $M = -\partial_x^2$, then the theory of elliptic functions can be used to build periodic solutions, having the general form of the Jacobian elliptic functions *snoidal*, *cnoidal*, *dnoidal* type. In case, $M = \mathcal{H}\partial_x$ and $p = 2$, Benjamin in ([11]) found a branch of travelling wave solutions φ_c with minimal period L for equation (4) when $c > \frac{2\pi}{L}$, having the form

$$\varphi_c(x) = \frac{4\pi}{L}\left(\frac{\sinh(\gamma)}{\cosh(\gamma) - \cos\left(\frac{2\pi x}{L}\right)}\right), \quad \tanh(\gamma) = \frac{2\pi}{cL}, \quad \gamma > 0.$$

A simple computation shows that the BO model (4) when $c > \frac{2\pi}{L}$, has a family of travelling wave solutions ψ_c with minimal period L having the mean zero property in $[0, L]$. In fact, if we define ψ_c as $\psi_c = \varphi_c - \rho$, where ρ is the average of φ_c on $[0, L]$, then ψ_c is solution of minimal period L of the (BO) model (4) with wave speed $\tilde{c} = c - 2\rho$, and having the mean zero property on $[0, L]$.

We will impose the same type of hypotheses adopted by Bona *et al.* in [2] to obtain sufficient and necessary conditions to guarantee stability/instability of L-periodic travelling wave solutions of the generalized Korteweg and de Vries equation (1) in a space having the mean zero property in $[0, L]$.

Assumptions on $\mathcal{L}_c$ and φ_c.
(H.1) There is an interval $(c_1, c_2) \subset [1, \infty)$ such that for every $c \in (c_1, c_2)$, there is a solution φ_c of (5). The curve $c \to \varphi_c$ is C^1 with values in $\mathcal{W}_L^{1+\frac{\mu}{2}}$.
(H.2) The operator $\mathcal{L}_c$ has a unique, negative, simple eigenvalue, with eigenfunction $\chi_c \in H_{per}^{1+\frac{\mu}{2}}([0, L])$. $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}_c$ with eigenfunction $\partial_x\varphi_c$, and the rest of the spectrum of $\mathcal{L}_c$ is positive and bounded away from zero. Moreover, The curve $c \to \chi_c$ is a continuous with values in $H_L^{1+\frac{\mu}{2}}$.

Now we precise the meaning of the stability and instability concept. For $\epsilon > 0$, consider the tube in $\mathcal{W}_L^{\frac{\mu}{2}}$

$$\mathcal{U}_\epsilon = \left\{u \in \mathcal{W}_L^{\frac{\mu}{2}} : \inf_r \|u - \tau_r\varphi_c\|_{H_{per}^{\frac{\mu}{2}}([0,L])} < \epsilon\right\},$$

where $\tau_r(f)(x) = f(r + x)$, $x \in \mathbb{R}$. This set is a neighborhood in $\mathcal{W}_L^{\frac{\mu}{2}}$ of the collection of all translates of φ_c.

Definition 2.2 *The periodic travelling wave φ_c is stable if and only if for $\epsilon > 0$ there exists $\eta > 0$ such that if $u_0 \in \mathcal{U}_\eta$, then $u(\cdot, t) \in \mathcal{U}_\epsilon$ for all $t \in \mathbb{R}$, where $u(\cdot, t)$ denotes the unique solution of the Cauchy problem associated with the generalized KdV type equation (1), with initial condition $u(\cdot, 0) = u_0(\cdot)$. We will say that the periodic travelling wave φ_c is unstable if φ_c is not stable.*

3 Adaptation of Grillakis, Shatah, Strauss Approach

In this section we adapt the Grillakis, Shatah, Strauss Approach, assuming that $\mathcal{L}_c$ and φ_c satisfy **(H.1)** and **(H.2)**. We start the discussion by proving a simple result regarding the operator $\mathcal{L}_c$, φ_c and χ_c, which will play an important role in our analysis. Hereafter, either p is an integer or $p = \frac{m_1}{m_2}$ where m_1, m_2 being positive integers with m_2 odd.

Lemma 3.1 *For $c > 1$ and $p > 1$,*

$$\langle \varphi_c, \chi_c \rangle \neq 0. \tag{9}$$

Proof. First we observe that a direct computation shows that

$$\langle \mathcal{L}_c(\varphi_c), \varphi_c \rangle = (1-p)\int_0^L \left(\varphi_c M\varphi_c + (c-1)\varphi_c^2\right)\, dx < 0. \tag{10}$$

On the other hand, we have that

$$\langle \varphi_c, \partial_x \varphi_c \rangle = 0.$$

Using this fact, we are able to decompose φ_c as $\varphi_c = \gamma\chi_c + \rho p_0$, for some p_0 in the positive subspace of $\mathcal{L}_c$ orthogonal to χ_c and $\partial_x\varphi_c$. In this case, $\langle \varphi_c, \chi_c \rangle = \gamma \neq 0$, otherwise $\varphi_c = \rho p_0$ and $\langle \mathcal{L}_c(\varphi_c), \varphi_c \rangle = \rho^2 \langle \mathcal{L}_c(p_0), p_0 \rangle \geq 0$, contradicting (10). □

Before we go further, we note that equation (6) gives a characterization of φ_c as a critical point of the functional E in $\mathcal{W}_L^1$ subject to the constraint $V(\varphi_c) = V(u)$, and so φ_c is either a saddle point or a local minimum. We will see as in [1, 2] that in the case $d''(c) < 0$, the function φ_c is a saddle point but not a local minimum, . We will see in next section that the function φ_c is critical point, which is in fact a local minimum, when $d''(c) > 0$.

In the coming result, using a slightly modification of the argument of Grillakis *et. al.* in ([1]), we are able to build a curve $\omega \to \varphi_\omega \in W_L^{1+\frac{\mu}{2}}$ satisfying conditions (1) and (2) described in the Introduction. Moreover, condition (3) regarding the existence of a Liapunov type functional is established in Lemma (3.4) below.

Theorem 3.2 *Let $c > 1$ and $p > 1$ be an integer. If $d''(c) < 0$, then there is a curve $\omega \to \psi_\omega \in \mathcal{W}_L^{1+\frac{\mu}{2}}$ which passes through φ_c, lies on the surface $V(u) = V(\varphi_c)$, and on which $E(u)$ has a strict local maximum at $u = \varphi_c$.*

Proof. For ω near c, we will see via the Implicit Function Theorem that there is a differentiable function $s(\omega)$ such that $s(c) = 0$.

Consider the function $(\omega, s) \to \psi_{(\omega,s)} \in W_L^{1+\frac{\mu}{2}}$ defined as $\psi_{(\omega,s)} = \varphi_\omega + s\varphi_c$. Note for $\omega = c$ and $s = 0$ that $\psi_{(c,0)} = \varphi_c$ and $V(\psi_{(c,0)}) = V(\varphi_c)$. We observe that

$$\frac{\partial}{\partial s} V(\psi_{(\omega,s)})\Big|_{\{s=0,\ \omega=c\}} = V'(\varphi_c)(\varphi_c) = \int_0^L \varphi_c^2(x)\, dx > 0.$$

From the Implicit Function Theorem, we assure the existence of a function $s(\omega)$ defined for ω near c. Now, we set $\psi_\omega := \psi_{(\omega,s(\omega))}$.

Then, since $\frac{d}{d\omega}V(\psi_\omega) = \langle V'(\psi_\omega), \frac{d}{d\omega}\psi_\omega \rangle = 0$, we conclude that

$$\frac{d}{d\omega}E(\psi_\omega) = \left\langle E'(\psi_\omega), \frac{d}{d\omega}\psi_\omega \right\rangle = \left\langle E'(\psi_\omega) + \omega V'(\psi_\omega), \frac{d}{d\omega}\psi_\omega \right\rangle$$

Using this and that $\int_0^L \psi_\omega(x)\,dx = 0$, we have that

$$\begin{aligned}
\frac{d^2}{d\omega^2}E(\psi_\omega) &= \frac{d^2}{d\omega^2}\left(E(\psi_\omega) + \omega V(\psi_\omega)\right) \\
&= \left\langle E'(\psi_\omega) + \omega V'(\psi_\omega), \frac{d^2}{d\omega^2}\psi_\omega \right\rangle \\
&+ \left\langle \left(E''(\psi_\omega) + \omega V''(\psi_\omega)\right)\left(\frac{d\psi_\omega}{d\omega}\right), \left(\frac{d\psi_\omega}{d\omega}\right) \right\rangle \\
&= A\int_0^L \frac{d^2\psi_\omega(x)}{d\omega^2}\,dx \\
&+ \left\langle \left(E''(\psi_\omega) + \omega V''(\psi_\omega)\right)\left(\frac{d\psi_\omega}{d\omega}\right), \left(\frac{d\psi_\omega}{d\omega}\right) \right\rangle \\
&= \left\langle \left(E''(\psi_\omega) + \omega V''(\psi_\omega)\right)\left(\frac{d\psi_\omega}{d\omega}\right), \left(\frac{d\psi_\omega}{d\omega}\right) \right\rangle.
\end{aligned}$$

Evaluating this at $\omega = c$, we have that

$$\frac{d^2}{d\omega^2}E(\psi_\omega)|_{\{\omega=c\}} = \langle \mathcal{L}_c y, y \rangle, \quad \text{where} \quad y = \frac{d\psi_\omega}{d\omega}\Big|_{\{\omega=c\}} = \frac{d\varphi_c}{dc} + s'(c)\varphi_c.$$

Using that $V(\varphi_c) = V(\psi_\omega)$, we have that

$$\begin{aligned}
0 = \frac{d}{d\omega}V(\psi_\omega)|_{\{\omega=c\}} = \int_0^L y\varphi_c\,dx &= \left\langle \frac{d}{dc}\varphi_c, \varphi_c \right\rangle + s'(c)\langle \varphi_c, \varphi_c \rangle \\
&= \frac{d}{dc}V(\varphi_c) + s'(c)\langle \varphi_c, \varphi_c \rangle
\end{aligned}$$

Thus we then have that

$$\int_0^L y\varphi_c\,dx = 0, \quad \text{and} \quad d''(c) = \frac{d}{dc}V(\varphi_c) = -s'(c)\langle \varphi_c, \varphi_c \rangle < 0. \tag{11}$$

Note that in particular, $s'(c) > 0$. On the other hand,

$$\mathcal{L}_c y = \mathcal{L}_c\left(\frac{d\varphi_c}{dc}\right) + s'(c)\mathcal{L}_c(\varphi_c) = -\varphi_c + s'(c)\mathcal{L}_c(\varphi_c).$$

Since $\langle y, \varphi_c\rangle = 0$, conclude that

$$\begin{aligned}\langle \mathcal{L}_c y, y\rangle &= s'(c)\left\langle \mathcal{L}_c(\varphi_c), y\right\rangle \\ &= s'(c)\left(\left\langle \mathcal{L}_c(\varphi_c), \frac{d\varphi_c}{dc}\right\rangle + s'(c)\left\langle \mathcal{L}_c(\varphi_c), \varphi_c\right\rangle\right) \\ &= -s'(c)\left\langle \varphi_c, \varphi_c\right\rangle + (s'(c))^2\left\langle \mathcal{L}_c(\varphi_c), \varphi_c\right\rangle \\ \\ &= d''(c) + (s'(c))^2\left\langle \mathcal{L}_c(\varphi_c), \varphi_c\right\rangle.\end{aligned}$$

But we show in previous Lemma that $\langle \mathcal{L}_c(\varphi_c), \varphi_c\rangle < 0$, for that $c > 1$ and $p > 1$ (see 10). In other words, we have established that

$$\frac{d^2}{d\omega^2}E(\psi_\omega)|_{\{\omega=c\}} = \langle \mathcal{L}_c y, y\rangle < 0, \quad c > 1, \quad p > 1,$$

as desired. □

Hereafter we only present the analogous result for the periodic case obtained by Grillakis *et. al* in [1] (or by Bona *et. al.* in [2]), pointing out the appropriate changes.

Lemma 3.3 *There exist $\epsilon > 0$ and a unique C^1 map $\alpha : \mathcal{U}_\epsilon \to \mathbb{R}/L$ such that*

1. $\langle u(\cdot + \alpha(u), \partial_x\varphi_c\rangle = 0,$
2. $\alpha(u(\cdot + r)) = \alpha(u) - r$ *modulo the period,*
3. $\alpha(\varphi_c) = 0,$
4. $\alpha'(u) = \frac{\partial_x\varphi_c(\cdot - \alpha(u))}{\int_0^L u(x)\partial_x^2\varphi_c(x - \alpha(u))\,dx}.$

Recall that we established for $c > 1$ and $p > 1$ the existence of y such that

$$\langle \mathcal{L}_c y, y\rangle < 0.$$

Using this fact, we are able to establish the following result.

Theorem 3.4 *Let $\mathcal{B}$ the functional defined as*

$$\mathcal{B}(u) = y(\cdot - \alpha(u)) - \frac{\langle u, y(\cdot - \alpha(u))\rangle}{\langle u, \partial_x^2\varphi_c(\cdot - \alpha(u))\rangle}\partial_x^2\varphi_c(\cdot - \alpha(u)).$$

Then $\mathcal{B}$ is a C^1 function from $\mathcal{U}_\epsilon$ into $\mathcal{W}_L^{\frac{\mu}{2}}$ such that commutes with translations, $\mathcal{B}(\varphi_c) = y$, and $\langle \mathcal{B}(u), u\rangle = 0$, for $u \in \mathcal{U}_\epsilon$.

The only comment is that $\mathcal{B}(u)$ has the mean zero property in $[0, L]$. In fact, since $y \in \mathcal{W}_L^{\frac{\mu}{2}}$,

$$\int_0^L \mathcal{B}(u)(x)\,dx = \int_0^L y(x-\alpha(u))\,dx - \frac{\langle u, y(\cdot-\alpha(u))\rangle}{\langle u, \partial_x^2\varphi_c(\cdot-\alpha(u))\rangle}\partial_x\varphi_c(\cdot-\alpha(u))\Big|_0^L = 0.$$

As done in Grillakis *et. al* paper [1] (or Bonna *et. al* ([2])), we may consider the initial value problem

$$\begin{cases} \frac{du}{d\lambda} &= \mathcal{B}(u), \\ u(0) &= v \in \mathcal{U}_\epsilon. \end{cases} \tag{12}$$

Corollary 3.5 *If $u = R(\lambda, v)$ denotes the solution of the initial value problem (12), then we have*

1. *R is a C^1 function for $|\lambda| < \lambda_0(v)$ for any $v \in \mathcal{U}_\epsilon$,*
2. *R commutes with translations for each λ,*
3. *$V(R(\lambda, v))$ is independent of λ, and*
4. *$\frac{\partial R}{\partial \lambda}(0, \varphi_c) = y$.*

We want to point out that the proofs of the previous result and the three coming results are the same as those in Grillakis *et. al.* work ([1]), except for the last one in which we have to use the particular property of functions φ_c and χ_c established in Lemma 3.1.

Lemma 3.6 *There is a C^1 function $\Lambda : \{v \in \mathcal{U}_\epsilon : V(v) = V(\varphi_c)\} \to \mathbb{R}$ such that*

$$E(R(\lambda, v)) > E(\varphi_c)$$

for all $v \in \mathcal{U}_\epsilon$ such that $v \in \mathcal{U}_\epsilon$ and $v \notin \{\varphi_c(\cdot + s) : s \in \mathbb{R}\}$.

Lemma 3.7 *For $v \in \mathcal{U}_\epsilon$ such that $V(v) = V(\varphi_c)$ and $v \notin \{\varphi_c(\cdot + s) : s \in \mathbb{R}\}$ we have*

$$E(\varphi_c) < E(v) + \Lambda(v)\langle E'(v), \mathcal{B}(v)\rangle.$$

Lemma 3.8 *The curve ϕ_ω satisfies $E(\phi_\omega) < E(\varphi_c)$ for $\omega \neq c$, $V(\phi_\omega) = V(\varphi_c)$ and $\Lambda(v)\langle E'(v), \mathcal{B}(v)\rangle$ changes sign as ω passes through c.*

Proof. From the proof of Theorem (3.2), we have that $y = \frac{d}{dc}\varphi_c + s'(c)\varphi_c$, with $s'(c) > 0$. Then as in either [1] or [2], we only need to assure that $\langle G'(\varphi_c)y, \chi_c\rangle \neq 0$. But we have that

$$\begin{aligned} \langle G'(\varphi_c)y, \chi_c\rangle &= \langle y, \chi_c\rangle \\ &= \left(-\frac{1}{\lambda_0} + s'(c)\right)\langle \varphi_c, \chi_c\rangle. \end{aligned}$$

Since $\lambda_0 < 0$, $s'(c) > 0$, and from (9), we conclude that $\langle G'(\varphi_c)y, \chi_c\rangle \neq 0$.

□

3.1 Stability and instability of periodic travelling waves

Now we are in position to establish the necessary results to show instability and stability, which are an adaptation of the work by Grillakis *et. al.* ([1]) or by Bonna *et. al.* in ([2]). Recall that we showed that φ_c is a saddle point but not a local minimum, in case for $d''(c) < 0$. Moreover, as done in Grillakis *et al.* ([1]-Theorem 4.7) we have that

Theorem 3.9 *Let $c > 1$ be fixed. If $d''(c) < 0$, then the φ_c- orbit is $\mathcal{W}_L^{\frac{\mu}{2}}$ -unstable with respect to the flow of equation (1).*

Now we will discuss the stability issue. We start establishing that φ_c is a critical point which is in fact local minimum, when $d''(c) > 0$.

Theorem 3.10 *Let $c > 1$ be fixed. If $d''(c) > 0$, then there is a constant $C > 0$ and $\epsilon > 0$ such that*

$$E(u) - E(\varphi_c) \geq C\|u(\cdot + \alpha(u)) - \varphi_c\|_{H_{per}^{\frac{\mu}{2}}}$$

for all $u \in U_\epsilon$ which satisfy $V(u) = V(\varphi_c)$.

Proof. For the sake of completeness we include the proof as in ([1]).

If $\langle w, \varphi_c\rangle = \langle w, \partial_x\varphi_c\rangle = 0$, then $w = \alpha_1\chi + \gamma_1 p_1$, where p_1 is in the positive subspace of $\mathcal{L}_c$. On the other hand, we have the following decomposition for $\partial_c\varphi_c$,

$$\partial_c\varphi_c = \alpha_0\chi + \beta_0\partial_x\varphi_c + \gamma_0 p_0.$$

Using that $\sqrt{\mathcal{L}_c}$ defines an inner product in the positive subspace of $\mathcal{L}_c$, we have that

$$\langle \mathcal{L}_c w, w\rangle = \alpha_1^2\lambda_0 + \gamma_1^2 \langle \mathcal{L}_c p_1, p_1\rangle \geq \alpha_1^2\lambda_0 + \gamma_1^2 \frac{\langle \mathcal{L}_c p_1, p_0\rangle^2}{\langle \mathcal{L}_c p_0, p_0\rangle}. \tag{13}$$

On the other hand,

$$0 = \langle -\varphi_c, w\rangle = \langle \mathcal{L}_c\partial_c\varphi_c, w\rangle = \alpha_0\alpha_1\lambda_0 + \gamma_0\gamma_1 \langle \mathcal{L}_c p_0, p_1\rangle. \tag{14}$$

But we also have that

$$d''(c) = \langle \varphi_c, \partial_c\varphi_c\rangle = -\langle \mathcal{L}_c\partial_c\varphi_c, \partial_c\varphi_c\rangle = -\alpha_0^2\lambda_0 - \gamma_0^2 \langle \mathcal{L}_c p_0, p_0\rangle > 0.$$

Then we obtain that

$$0 < \gamma_0^2 \langle \mathcal{L}_c p_0, p_0\rangle < -\alpha_0^2\lambda_0. \tag{15}$$

Using (14) and (15) in inequality (13), we conclude that

$$\langle \mathcal{L}_c w, w\rangle > \alpha_1^2\lambda_0 + \left(\frac{\alpha_0^2\alpha_1^2\lambda_0^2}{\gamma_0^2}\right)\left(\frac{\gamma_0^2}{-\alpha_0^2\lambda_0}\right) = 0. \tag{16}$$

We have already shown that if Π denotes the orthogonal projection onto $[\partial_x\varphi_c]^\perp$, then there exists a positive constant δ such that for $w \in H_{per}^{\frac{\mu}{2}}$ with $\langle \varphi_c, w\rangle = 0$,

$$\langle \mathcal{L}_c w, w\rangle \geq \delta \|\Pi w\|^2_{H^{\frac{\mu}{2}}_{per}}. \tag{17}$$

Let $u \in \mathcal{U}_\epsilon$ and consider $z = u(\cdot + \alpha(u)) - \rho\varphi_c$, where ρ is chosen such that $\langle z, \varphi_c\rangle = 0$. Now, by (3.3)-(i) and using that φ_c and $\partial_x\varphi_c$ are orthogonal, we conclude that $\langle z, \partial_x\varphi_c\rangle = 0$. In other words, z is orthogonal to φ_c and $\partial_x\varphi_c$. As a consequence of this,

$$\langle \mathcal{L}_c z, z\rangle \geq \delta \|z\|^2_{H^{\frac{\mu}{2}}_{per}}. \tag{18}$$

Now, using the translation invariance of V,

$$V(\varphi_c) = V(u) = V(u(\cdot + \alpha(u)).$$

Then by Taylor's Theorem we conclude that

$$\begin{aligned}
V(\varphi_c) &= V(u(\cdot + \alpha(u))) \\
&= V(\varphi_c) + \langle \varphi_c, u(\cdot + \alpha(u)) - \varphi_c\rangle + O\left((\|u(\cdot + \alpha(u)) - \varphi_c\|^2_{H^{\frac{\mu}{2}}_{per}} \right) \\
&= V(\varphi_c) + (\rho - 1)\|\varphi_c\|^2_{L^2_{per}} + O\left(\|u(\cdot + \alpha(u)) - \varphi_c\|^2_{H^{\frac{\mu}{2}}_{per}} \right).
\end{aligned}$$

In other words, we have that $\rho - 1 = O\left((\|u(\cdot + \alpha(u)) - \varphi_c\|^2_{H^{\frac{\mu}{2}}_{per}} \right)$. Using the same argument on the Taylor expansion for $\mathcal{F}$, $\mathcal{F}(u) = \mathcal{F}(u(\cdot + \alpha(u)))$

$$= \mathcal{F}(\varphi_c) + \frac{1}{2}\langle \mathcal{L}_c(u(\cdot + \alpha(u)) - \varphi_c), u(\cdot + \alpha(u)) - \varphi_c\rangle + O\left(\|v\|^2_{H^{\frac{\mu}{2}}_{per}} \right)$$

where $v = u(\cdot + \alpha(u)) - \varphi_c$. Moreover, $v = u(\cdot + \alpha(u)) - \varphi_c = (\rho - 1)\varphi_c + z$.

Then, last equality implies for $||v||_{H^{\frac{\mu}{2}}_{per}}$ small enough that

$$\begin{aligned}
E(u) - E(\varphi_c) &= \frac{1}{2}\langle \mathcal{L}_c(u(\cdot + \alpha(u)) - \varphi_c), u(\cdot + \alpha(u)) - \varphi_c\rangle + O\left(\|v\|^2_{H^{\frac{\mu}{2}}_{per}} \right) \\
&= \frac{1}{2}\langle \mathcal{L}_c(v), v\rangle + O\left(\|v\|^2_{H^{\frac{\mu}{2}}_{per}} \right) \\
&= \frac{1}{2}\langle \mathcal{L}_c(z), z\rangle + O((\rho - 1)^2) + O((\rho - 1)\|v\|_{H^{\frac{\mu}{2}}_{per}}) \\
&+ O\left(\|v\|^2_{H^{\frac{\mu}{2}}_{per}} \right) \\
&\geq \frac{\delta}{2}\|z\|^2_{H^{\frac{\mu}{2}}_{per}} + O\left(\|v\|^2_{H^{\frac{\mu}{2}}_{per}} \right) \\
&\geq \frac{\delta}{2}\left[\|v\|_{H^{\frac{\mu}{2}}_{per}} - |\rho - 1|\|\varphi_c\|_{H^{\frac{\mu}{2}}_{per}} \right]^2 + O\left(\|v\|^2_{H^{\frac{\mu}{2}}_{per}} \right) \\
&\geq \frac{\delta}{4}\|v\|^2_{H^{\frac{\mu}{2}}_{per}}.
\end{aligned}$$

□

Finally we are able to establish the stability result.

Theorem 3.11 *The periodic travelling wave φ_c is stable if and only if $d''(c) > 0$.*

Proof. Suppose that $d''(c) > 0$ and assume that φ_c is unstable.

Let $\{(u_{0,n})_n\} \subset \mathcal{W}_L^{\frac{\mu}{2}}$ be any sequence such that

$$\lim_{n\to\infty}\left[\inf_r \|u_{0,n} - \varphi_c(\cdot + r)\|_{H_{per}^{\frac{\mu}{2}}}\right] = 0.$$

For each n, let u_n denote the unique solution of the Cauchy problem associated with the generalized KdV equation (1) with initial condition $u_{0,n}$. Take $(t_n)_n$ such that, for each n we have that $u_n(\cdot, t_n) \in \partial\mathcal{U}_{\frac{1}{2}\epsilon}$. Due to the translation invariance of the continuous functionals E and V, we conclude that,

$$E(u_n(\cdot, t_n)) = E(u_{0,n}) \to E(\varphi_c), \quad \text{and} \quad V(u_n(\cdot, t_n)) = V(u_{0,n}) \to V(\varphi_c).$$

But we are able to choose $w_n \in \mathcal{U}_\epsilon$ such that $\|w_n - u_n(\cdot, t_n)\|_{\mathcal{W}^1} \to 0$.
Then by Theorem (3.10) we conclude that

$$\|w_n - \varphi_c(\cdot, \alpha(w_n))\|^2_{H_{per}^{\frac{\mu}{2}}} = \|w_n(\cdot, \alpha(w_n)) - \varphi_c\|^2_{H_{per}^{\frac{\mu}{2}}}$$

$$\leq \frac{1}{C}(E(w_n) - E(\varphi_c)) \to 0.$$

This fact implies that,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|w_n - u_n(\cdot, t_n)\|_{H_{per}^{\frac{\mu}{2}}} + \lim_{n\to\infty} \|w_n - \varphi_c(\cdot, \alpha(w_n))\|_{H_{per}^{\frac{\mu}{2}}} \\ &\geq \lim_{n\to\infty} \|u_n(\cdot, t_n) - \varphi_c(\cdot, \alpha(w_n))\|_{H_{per}^{\frac{\mu}{2}}} \geq 0. \end{aligned}$$

In other words, $u_n(\cdot, t_n)$ tends to the orbit of φ_c, proving by contradiction that φ_c is stable. The other direction follows by noting that the set $\{c > 1 : \varphi_c\}$ is stable is an open set and by using Theorem (3.9).

□

Acknowledgments: J. Quintero was supported by Universidad del Valle, Cali, Colombia. This work was completed and revised while JRQ was on sabbatical (01-2009/01-2010).

References

[1] Grillakis, M., Shatah, J., and W. Strauss. (1987). Stability Theory of Solitary Waves in Presence of Symmetry I., J. Functional Analysis. **74**, 160–197.

[2] Bona, J., Souganidis, P. E., and Strauss, W. A. (1987). Stability and Instability of Solitary Waves of Korteweg-de Vries Type. Proc. R. Soc. London A. **411**, 395–412.

[3] Korteweg, D. J., and de Vries, G. (1895). On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves. Phil. Mag. **39**, (5) 422–443.

[4] Angulo, J., Bona, J., and Scialom, M. (2006). Stability of Cnoidal Waves, Adv. Diff. Equations. **11**, 1321-1374.

[5] Deconinck, B. and Bottman, N. (2008). KdV Cnoidal waves are linearly stable. preprint.

[6] Deconinck, B and Kapitula, T (2009). On orbital (in)stability of spatially Periodic stationary solutions of generalized Korteweg de Vries Equations. Preprint.

[7] Johnson, M. (2009) Nonlinear stability of periodic traveling wave solutions of the generalized Korteweg-de Vries equation. preprint.

[8] Gallay, T. and Hărăgus, M. (2007). Orbital stability of periodic waves for the nonlinear Schrödinger equation. J. Dyn. Diff. Eqns. **19**. 825865.

[9] Angulo, J., and Natali, F. (2008). Positivity Porperties and Stability of Periodic Travalling-Waves Solutions. Preprint.

[10] Gardner, R. (1997). Spectral Analysis of Long Wavelenght Periodic Waves and a Applications. J. für Die Reine und Angewandte Mathematik. **491**. 149–181.

[11] Benjamin, T. B. (1974). Lecture en Linear Wave Motion, Nonlinear Wave Motion, AC. Newell, ed. AMS, Providence, RI **15**, 3–47.

[12] Bona, J., and Smith, R. (1975). The Initial Value Problem for Korteweg-de Vries Equations. Philos. Trans. Toyal Soc. London. **Ser. A. 278**, 555–601.

[13] Bona, J., and Scott, R. (1976). Solutions of the Korteweg-de Vries equations in Fractional Order Sobolev Spaces. Duke Math J. **43** 87–99.

[14] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., and Tao, T. (2003). Sharp Global Well-Poseness for KDV and Modified KDV on $\mathbb{R}$ and T. J. American Math. Soc.. **16**, 705–749.

[15] Kato, T. (1979). On the Korteweg-de Vries equation. Lectures Notes in Mathematics. **448**, 25–70.

[16] Kato, T. (1983). On the Cauchy Problem for the (generalized) Korteweg-de Vries Equations. Studies in Applied Math., advances in Mathematics Suppl. Studies. **8**, 93–128.

[17] Ince, E. L. (1940). The periodic Lamé function. Proc. Roy. Soc. **60**, 47–63.

[18] Magnus, W., and Winkler, S. (1976) *Hill's Equation. Tracts in Pure and Appl. Math.* **20**, Wesley. N.Y.

Author's address

José R. Quintero
Departamento de Matemáticas, Universidad del Valle, Cali - Colombia
quinthen@univalle.edu.co