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ABOUT THE UNIQUENESS OF CONFORMAL METRICS WITH PRESCRIBED SCALAR AND MEAN CURVATURES ON COMPACT MANIFOLDS WITH BOUNDARY

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Abstract

Let (M^n, g) be an $n-$dimensional compact Riemannian manifold with boundary with $n \geq 2$. In this paper we study the uniqueness of metrics in the conformal class of the metric g having the same scalar curvature in M, ∂M, and the same mean curvature on the boundary of M, ∂M. We prove the equivalence of some uniqueness results replacing the hypothesis on the first Neumann eigenvalue of a linear elliptic problem associated to the problem of conformal deformations of metrics for one about the first Dirichlet eigenvalue of that problem.

Keywords: Conformal metrics, scalar curvature, mean curvature.

1 Introduction

Let (M^n, g) be an $n-$dimensional compact Riemannian manifold with boundary. Let R_g denote its scalar curvature and H_g the trace of the second fundamental form. We let $h_g = \frac{H_g}{n-1}$ be the mean curvature of the boundary of M, ∂M. In [1] and [3] it has been studied to what extent the scalar curvature and the mean curvature of the boundary determine the metric within its conformal class, where the conformal class of a metric g, denoted by $[g]$, is the set of metrics of the form φg where φ is a smooth positive function defined on M.

When $n = 2$ and $\widetilde{g} = e^{2u} g$ the function u satisfies the following non-linear elliptic equation:

$$\begin{cases} \Delta_g u - K_g + K e^{2u} = 0 & \text{in} \quad M, \\ \dfrac{\partial u}{\partial \eta_g} + k_g - k_{\widetilde{g}} e^{u} = 0 & \text{on} \quad \partial M, \end{cases} \tag{1}$$

where $K_g = \frac{R_g}{2}$ and $k_g = h_g$ denote the Gaussian curvature and the geodesic curvature of M with respect to the metric g.
If $n \geq 3$ and $\widetilde{g} = u^{\frac{4}{n-2}} g$ then the function u satisfies the non-linear elliptic equation:

$$\begin{cases} \Delta_g u - c(n) R_g u + c(n) R_{\widetilde{g}} u^{\frac{n+2}{n-2}} = 0 & \text{in} \quad M, \\ \dfrac{\partial u}{\partial \eta} + \frac{n-2}{2} h_g u - \frac{n-2}{2} h_{\widetilde{g}} u^{\frac{n}{n-2}} = 0 & \text{on} \quad \partial M, \end{cases} \tag{2}$$

where $c(n) = \frac{n-2}{4(n-1)}$.

In [1] and [3] the following question is investigated: Given $\widetilde{g} \in [g]$ with $R_g = R_{\widetilde{g}}$ in M, and $h_g = h_{\widetilde{g}}$ on M, when is $\widetilde{g} = g$? This geometric question is equivalent to the following uniqueness questions in PDEs: When $n = 2$ assume that u is the solution of problem (1) where $K_g = K_{\widetilde{g}}$ and $k_g = k_{\widetilde{g}}$, is the function u the constant function 0? If $n \geq 3$ and u is the solution of problem (1) where $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$, is the function u the constant function 1?
We observe that if $R_g = R_{\widetilde{g}} = 0$, $h_g = h_{\widetilde{g}} = 0$ then $g = \gamma \widetilde{g}$, where γ is a positive constant. From now on we assume that the functions R_g and h_g do not vanish simultaneously.

Let us introduce the operator (L_1, B_1) defined by

$$\begin{cases} L_1 = \Delta_g + \frac{R_g}{n-1} & \text{in} \quad M, \\ B_1 = \frac{\partial}{\partial \eta} - h_g & \text{on} \quad \partial M. \end{cases} \tag{3}$$

Let denote by λ and by β the first Dirichlet eigenvalue and the first Neumann eigenvalue of the operator (L_1, B_1), respectively. Let the function $\widetilde{f}$ be a first positive Dirichlet eigenfunction of the operator (L_1, B_1), that is $\widetilde{f}$ satisfies the boundary value problem

$$\begin{cases} L_1(\widetilde{f}) + \lambda \widetilde{f}) = 0 & \text{in} \quad M, \\ B_1(\widetilde{f})) = 0 & \text{on} \quad \partial M. \end{cases} \tag{4}$$

Now, let the function f be a first positive Neumann eigenfunction of the operator (L_1, B_1), that is, f satisfies the boundary value problem

$$\begin{cases} L_1(f) = 0 & \text{in} \quad M, \\ B_1(f) = \beta f & \text{on} \quad \partial M. \end{cases} \tag{5}$$

Escobar in [1] proved the following uniqueness theorem

Theorem 1. *Let (M^n, g) be a compact Riemannian manifold with boundary and $h_g \leq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ and* and *$h_g = h_{\widetilde{g}}$. If both λ and $\widetilde{\lambda}$ are positive or one of them is equal to zero then $\widetilde{g} = g$.*

The following proposition follows from this theorem and the variational characterization of the first Dirichlet eigenvalue.

Proposition 1. *Let (M^n, g) be a compact Riemannian manifold with boundary. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}} \leq 0$ and $h_g = h_{\widetilde{g}} \leq 0$. Then $\widetilde{g} = g$.*

In [3] we found a result which is similar to Escobar's uniqueness theorem (Theorem 1); in our result we replace Escobar's hypothesis of non-positive mean curvature by non-negative scalar curvature.

Theorem 2. *Let (M^n, g) be a compact Riemannian manifold with boundary and $R_g \geq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. If both λ and $\widetilde{\lambda}$ are positive or one of them is equal to zero then $\widetilde{g} = g$.*

In [3], as a consequence of Theorems 1 and 2 we found, respectively, the following results

Theorem 3. *Let (M^n, g) be a compact Riemannian manifold with boundary and $R_g \geq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. If both β and $\widetilde{\beta}$ are positive or one of them is equal to zero then $\widetilde{g} = g$.*

Theorem 4. *Let (M^n, g) be a compact Riemannian manifold with boundary and $h_g \leq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. If both β and $\widetilde{\beta}$ are positive or one of them is equal to zero then $\widetilde{g} = g$.*

In the next section we give direct proofs of Proposition 1 and Theorems 3 and 4. From the variational characterizations of the eigenvalues λ and β it follows that $\lambda \geq 0$ if and only if $\beta \geq 0$, and $\lambda = 0$ if and only if $\beta = 0$. This fact and Theorems 3 and 4 yield, respectively, to the Theorems 1 and 2, showing the equivalence of such results.

2 Uniqueness theorems

First we give a proof of Proposition 1.
Proof . First, let us consider the case $n \geq 3$. Set $\widetilde{g} = u^{\frac{4}{n-2}} g$ and $v = u^{\frac{-2}{n-2}} - 1$.
A straightforward calculation shows that

$$\begin{cases} \Delta v + \dfrac{R_g}{2(n-1)} v(u^{\frac{2}{n-2}} + 1) = \dfrac{2n}{(n-2)^2} u^{-\frac{2(n-1)}{n-2}} |\nabla u|^2 \geq 0 & \text{in} \quad M, \\ \dfrac{\partial v}{\partial \eta} = h_g v & \text{on} \quad \partial M. \end{cases} \tag{6}$$

Using the hypothesis $R_g \leq 0$, we get $\dfrac{R_g}{2(n-1)}(u^{\frac{2}{n-2}} + 1) \leq 0$.

Let $v(x_0) = \max\{v(x) : x \in M\}$. If $x_0 \in \partial M$ and $v(x_0) < 0$ then $v < 0$, $u > 1$ and $\widetilde{g} > g$.
If $x_0 \in \partial M$ and $v(x_0) \geq 0$ then Hopf's lemma implies that $\dfrac{\partial v(x_0)}{\partial \eta} > 0$ or v is a non-negative constant. The inequality $\dfrac{\partial v(x_0)}{\partial \eta} > 0$ is impossible because of the hypothesis $h_g \leq 0$ and the inequality

$$\frac{\partial v(x_0)}{\partial \eta} = h_g(x_0) v(x_0) \leq 0.$$

Hence v is a non-negative constant. From here and the equation (6) we obtain

$$\frac{2n}{(n-2)^2} u^{-\frac{2(n-1)}{n-2}} |\nabla u|^2 = 0.$$

This equation and the fact that $u > 0$ implies that $\nabla u = 0$. It follows that u is a constant. From (2), using thatR_g and h_g do not vanish simultaneously we conclude that $u = 1$ and $\widetilde{g} = g$.
On the other hand, if $x_0 \in M \setminus \partial M$ the maximum principle implies that $v < 0$ or v is a non negative constant. If $v < 0$ we conclude that $u > 1$ and $\widetilde{g} > g$. If v is a non-negative then, as before, we get $u = 1$ and $\widetilde{g} = g$. Hence, we have obtained for $n \geq 3$ that $\widetilde{g} > g$ or $\widetilde{g} = g$ Now consider the case $n \geq 2$. Let $\widetilde{g} = e^{2u} g$ and $v = e^{-u} - 1$.$v = e^{-u} - 1$.
Then v satisfies

$$\begin{cases} \Delta v + K_g v(1+e^u) = e^{-u}|\nabla u|^2 \geq 0 & \text{in} \quad M, \\ \dfrac{\partial v}{\partial \eta} = k_g v & \text{on} \quad \partial M. \end{cases} \tag{7}$$

Using the hypothesis $R_g \leq 0$ we get $K_g(1+e^u) \leq 0$. Arguing as in the case of dimension $n \geq 3$, we get again that $\widetilde{g} = g$ or $\widetilde{g} > g$. In any case we have obtained $\widetilde{g} = g$ or $\widetilde{g} > g$; changing the roles of $\widetilde{g}$ and g, we also get $g > \widetilde{g}$ or $g = \widetilde{g}$ and we conclude that $\widetilde{g} = g$.

Lemma 1. *Let (M^n, g) be a compact Riemannian manifold with boundary and $R_g \geq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. If $\beta = 0$ then $\widetilde{g} = g$ and if $\beta > 0$ then $\widetilde{g} = g$ or $\widetilde{g} > g$.*

Proof . Let us consider first the case $n \geq 3$ Set $\widetilde{g} = u^{\frac{4}{n-2}} g$ and $v = u^{\frac{-2}{n-2}} - 1$. A straightforward calculation shows that

$$\begin{cases} \Delta v = \dfrac{2n}{(n-2)^2} u^{-\frac{2(n-1)}{n-2}} |\nabla u|^2 - \dfrac{R_g}{2(n-1)} v(u^{\frac{2}{n-2}} + 1) & \text{in} \quad M, \\ \dfrac{\partial v}{\partial \eta} = h_g v & \text{on} \quad \partial M. \end{cases} \tag{8}$$

Let f be a positive eigenfunction associated to the first Neumann eigenvalue of the operator (L_1, B_1). Thus f is a solution of the boundary value problem (5). By setting $w = \dfrac{v}{f}$, since $R_g \geq 0$ we get

$$\frac{w R_g}{2(n-1)} \left(1 - u^{\frac{2}{n-2}}\right) = \frac{u^{-\frac{2}{n-2}} R_g}{2(n-1) f} \left(1 - u^{\frac{2}{n-2}}\right)^2 \geq 0, \tag{9}$$

and therefore

$$\begin{cases} \Delta w + \frac{2}{f} \nabla f . \nabla w = \frac{2n}{(n-2)^2 f} u^{-\frac{2(n-1)}{n-2}} |\nabla u|^2 + \frac{u^{-\frac{2}{n-2}} R_g}{2(n-1) f} \left(1 - u^{\frac{2}{n-2}}\right)^2 \geq 0 & \text{in } M, \\ \dfrac{\partial w}{\partial \eta} = -w\beta & \text{on } \partial M. \end{cases} \tag{10}$$

Let $w(x_0) = \max\{w(x) : x \in M\}$ and let us assume $\beta = 0$. If $x_0 \in \partial M$, since $\dfrac{\partial w(x_0)}{\partial \eta} = 0$, by Hopf's lemma we get $w(x_0) < 0$ or w

is a non-negative constant.

If $w(x_0) < 0$ then $\widetilde{g} > g$. If w is a non-negative constant, by equation (8), we get

$$\frac{2n}{(n-2)^2} u^{-\frac{2(n-1)}{n-2}} |\nabla u|^2 + \frac{u^{-\frac{2}{n-2}} R_g}{2(n-1)} (1 - u^{\frac{2}{n-2}})^2 = 0. \tag{11}$$

Hence $\nabla u = 0$ and u is a constant. From (2), using that R_g and h_g do not vanish simultaneously we conclude that $u = 1$ and $\widetilde{g} = g$. If $x_0 \in M \setminus \partial M$ the maximum principle implies that w is a constant, and therefore we get the equation (10) again and as before we conclude that $\widetilde{g} = g$.
Now we consider $\beta > 0$. When $x_0 \in \partial M$, if $w(x_0) < 0$ then v is negative, $u > 1$ and $\widetilde{g} > g$. If $w(x_0) \geq 0$, Hopf's lemma implies that $\frac{\partial w}{\partial \eta}(x_0) > 0$ or w is a nonnegative constant. If

$$\frac{\partial w}{\partial \eta}(x_0) = -w(x_0)\beta > 0,$$

then $w(x_0) < 0$, which is a contradiction and we conclude that w is a nonnegative constant. If w is a nonnegative constant we obtain again the equation (11) and we conclude $\widetilde{g} = g$.

Next, we will consider the case $n = 2$ Set $\widetilde{g} = e^{2u} g$ and $v = e^{-u} - 1$. Then the function v satisfies the equations

$$\begin{cases} \Delta v = e^{-u} |\nabla u|^2 - K_g v (1 + e^u) & \text{in} \quad M, \\ \dfrac{\partial v}{\partial \eta} = k_g v & \text{on} \quad \partial M. \end{cases} \tag{12}$$

Let f be a positive eigenfunction associated to the first Neumann eigenvalue of the operator (L_1, B_1), in other words, f satisfies the boundary value problem (5). Therefore the function $w = \frac{v}{f}$ satisfies

$$\begin{cases} \Delta w + \dfrac{2}{f} \nabla f . \nabla w = \dfrac{e^{-u}}{f} |\nabla u|^2 + K_g \dfrac{e^{-u}}{f} (1 - e^u)^2 \geq 0 & \text{in} \quad M, \\ \dfrac{\partial w}{\partial \eta} = -\beta w & \text{on} \quad \partial M. \end{cases} \tag{13}$$

Arguing in the same way as in the case $n \geq 3$ we get again that $\widetilde{g} = g$ or $\widetilde{g} > g$.

Lemma 2. *Let (M^n, g) be a compact Riemannian manifold with boundary and $h_g \leq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. If $\beta = 0$ then $\widetilde{g} = g$ and if $\beta > 0$ then $\widetilde{g} = g$ or $\widetilde{g} > g$.*

Proof. To prove our lemma when $n \geq 3$, we let $\widetilde{g} = u^{\frac{4}{n-2}} g$ and $v = u^{\frac{-2}{n-2}} - 1$. A straightforward calculation shows that

$$\begin{cases} \Delta v = \dfrac{4(n+2)}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 - \dfrac{R_g}{n-1} v & \text{in} \quad M, \\ \dfrac{\partial v}{\partial \eta} = \dfrac{2h_g}{1 + u^{\frac{2}{n-2}}} v & \text{on} \quad \partial M. \end{cases} \tag{14}$$

As in the previous lemma, let f be a positive eigenfunction associated to the first Neumann eigenvalue of the operator (L_1, B_1). By setting $w = \dfrac{v}{f}$ we get

$$\begin{cases} f\Delta w + 2\nabla f.\nabla w = \dfrac{4(n+2)}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 \geq 0 & \text{in} \quad M, \\ \dfrac{\partial w}{\partial \eta} = \dfrac{h_g(1 - u^{\frac{2}{n-2}})}{1 + u^{\frac{2}{n-2}}} w - \beta w & \text{on} \quad \partial M. \end{cases} \tag{15}$$

Let $w(x_0) = \max\{w(x) : x \in M\}$. We have to consider two cases.
First, let us assume $\beta > 0$. If $x_0 \in \partial M$ and $w(x_0) < 0$ then $\widetilde{g} > g$. If $x_0 \in \partial M$ and $w(x_0) \geq 0$ then by Hopf's lemma we get $\dfrac{\partial w}{\partial \eta}(x_0) > 0$ or w is a nonnegative constant.
If $\dfrac{\partial w}{\partial \eta}(x_0)$ were positive, from the inequalities

$$0 < \frac{\partial w}{\partial \eta}(x_0) = \frac{h_g(1 - u^{\frac{2}{n-2}})}{1 + u^{\frac{2}{n-2}}} w(x_0) - \beta w(x_0), \tag{16}$$

and

$$h_g \frac{(1 - u^{\frac{2}{n-2}})}{1 + u^{\frac{2}{n-2}}} w \quad = \quad h_g \frac{u^{-\frac{4}{n-2}}(1 - u^{\frac{2}{n-2}})^2}{f} \leq 0, \tag{17}$$

we would get

$$-\beta w(x_0) > 0.$$

Hence $w(x_0) < 0$ which is a contradiction; consequently w is a nonnegative constant. Arguing as in the previous lemma we conclude that $u = 1$ and $\widetilde{g} = g$. If $x_0 \in M \setminus \partial M$ the maximum principle implies that w is a constant. As before we conclude that $\widetilde{g} = g$.
Now let us assume $\beta = 0$. If $x_0 \in \partial M$, using the equations (15) and the inequality (17) we get $\frac{\partial w}{\partial \eta}(x_0) \leq 0$. Hopf's lemma implies that $w < 0$ or w is a nonnegative constant and we conclude again that $\widetilde{g} = g$ or $\widetilde{g} > g$. If $x_0 \in M \setminus \partial M$ using the maximum principle we arrive to $\widetilde{g} = g$.
Now, we will consider the case $n = 2$ Set $\widetilde{g} = e^{2u}g$ and $v = e^{-u} - 1$. Then v satisfies:

$$\begin{cases} \Delta v = 4e^{-2u}|\nabla u|^2 - 2K_g v & \text{in} \quad M, \\ \dfrac{\partial v}{\partial \eta_g} = \dfrac{k_g v}{1 + e^u} & \text{on} \quad \partial M. \end{cases} \tag{18}$$

As before, let f be a positive eigenfunction associated to the first Neumann eigenvalue of the operator (L_1, B_1). Since f is a solution of the boundary value problem (5) then the function $w = \frac{v}{f}$ satisfies

$$\begin{cases} \Delta w + 2\nabla f.\nabla w = 4e^{-2u}|\nabla u|^2 & \text{in} \quad M, \\ \dfrac{\partial w}{\partial \eta_g} = wk_g\dfrac{1 - e^u}{1 + e^u} - \beta w & \text{on} \quad \partial M. \end{cases} \tag{19}$$

Arguing in the same way as in the case $n \geq 3$ we get again that $\widetilde{g} = g$ or $\widetilde{g} > g$.

Proof of Theorems 3 and 4. If either β or $\widetilde{\beta}$ vanishes then the previous lemmas yield to $\widetilde{g} = g$ If both β and $\widetilde{\beta}$ and $\widetilde{\beta}$ are positive, the previous lemmas imply that $\widetilde{g} = g$ or $\widetilde{g} > g$ and $\widetilde{g} = g$ or $\widetilde{g} < g$. Hence, the only possibility is $\widetilde{g} = g$.

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