# Equations of Lax type with several brackets

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**Abstract**

In the present paper we introduce a new hierarchy, each equation of which is defined by several brackets of two pseudodifferential operators. We deal with some algebraic properties of these equations. In particular, we show that, as in the classic case, they are related to the presence of an infinite sequence of first integrals. A new version of the Zakharov-Shabat type equations is proved. Formal solutions of this hierarchy are constructed using a "several brackets bilinear identity".

## 1    Introduction

Recently, great progress has been made in the construction of integrable systems. It has been found that integrable systems usually admit a natural extension by odd variables, and multi-dimensional and integral generalizations of soliton equation have been considered. In this paper we will concentrate on another form of construction of these systems: equations of Lax type with several brackets.

Equations with several brackets arose originally in the work of Brockett [2], [3], for ordinary differential equations, and then in Felipe [6] for partial differential equations of Lax type with double brackets (see [7] for the super Brockett equations type).

The study of flows of the form

$$\frac{\partial L}{\partial t} = [L, [L, N]] \tag{1}$$

on Lie algebras of finite dimension (that is for ordinary differential equations) was first made by R. W. Brockett (see [2] and [3]), who observed that this type of equations are interesting because of their relation with problems of counting and combinatorial optimization. In the case of the Toda lattices, we obtain an integral Hamiltonian system that is also a gradient flow. Also R. W. Brockett, A. Bolch

and T. S. Ratiu (see [1]) introduced a continuous analog of Toda Lattices (partial differential equations) that can be written as (1) and showed, in addition, that this equation has a set of functional invariants or conserved quantities.

More recently Felipe (see [6]) introduced a hierarchy on the Lie algebra of pseudodifferential operators (the so-called hierarchy of Brockett) of the following form:

$$\frac{\partial L}{\partial t} = \left[ L, \left[ L, (L^n)_+ \right] \right], \qquad n = 1, 2, 3, \ldots . \tag{2}$$

Remarkably, this hierarchy shares many properties of its finite dimensional counter part, and one of our goals is to extend this study to hierarchies defined by more brackets.

As yet another generalization, we investigate here equations of the form:

$$\frac{\partial L}{\partial t} = \underbrace{[L, [L, \cdots [L, P] \cdots ]}_{m-times} \tag{3}$$

where $L$, $P : \mathbb{R}_+ \to g$ and $g$ is a Lie algebra. We will show that the algebraic situation is the same as the usual $KP$ system studied in [5], [9] and [10] (See [11] for the super $KP$ equation). In particular, that a construction of Adler gives, in this context, invariant functions.

*As is well known, the definition of complete integrability in the infinite dimensional case is more subtle than in the finite dimensional one. The one we will use here is the existence of an infinite number of conserved quantities.*

This paper is organized as follows. In section 2 we generalize the usual $KP$ framework to the case of several brackets. Then, in section 3 we show that as in the classical case the equations (3) are related to the presence of an infinite sequence of first integrals. Section 4 exhibits some new integral systems. Finally, in section 5 we will emphasize the role of the Baker-Akhiezer function and one of our results will be a "several brackets bilinear identity".

## 2   Notations and Preliminary Results

In this section we give a brief introduction to pseudodifferential operators. Ordinary differential operators can be written as

$$P = p_0 \partial^n + p_1 \partial^{n-1} + \cdots + p_n$$

where the coefficients $p_i$ are supposed to lie in some differential algebra over $\mathbb{C}$ of smooth functions of $x$, for example $\mathbb{C}[[x]]$, and $\partial = \frac{d}{dx}$. A pseudodifferential operator is a formal series of the form:

$$R = \sum_{-\infty}^{n} r_i(x) \partial^i, \qquad n \in \mathbb{N}, \tag{4}$$

where the operator $\partial^i$, $i \in \mathbb{Z}$, acts according to the rules (see e.g. [5])

$$\partial^i f = f \partial^i + \binom{i}{1} f' \partial^{i-1} + \cdots, \qquad \binom{i}{j} = \frac{i(i-1)\cdots(i-j+1)}{j!}.$$

Notice that the above equation is a generalization of the well-known Leibniz rule.

We introduce the product of two operators of the form (4). First we define

$$\partial^n \cdot \partial^m = \partial^{n+m}, \ \partial^0 = 1 \qquad (n, m \in \mathbb{Z}).$$

It is easy to see that

$$\partial^{-1} r = \sum_{j=0}^{\infty} (-1)^j r^{(j)} \partial^{-(j+1)}$$

where $r^{(j)} = \frac{\partial^j r}{\partial x^j}$ (see e.g. [5]).

It is easy to check that this makes the set of all pseudodifferential operators an associative algebra, which we denote by $\Psi DO$. If $R = \sum r_i \partial^i$ is a pseudodifferential operator, we will write $R_+$ for the differential operator part, $R_+ = \sum_{i \geq 0} r_i \partial^i$, and $R_- = \sum_{i < 0} r_i \partial^i$.

Thus $R = R_+ + R_-$. An element $R \in \Psi DO$ has order $n$ if

$$R = \sum_{-\infty}^{n} r_i \partial^i$$

and $r_n \neq 0$. We denote $ord(R) = n$. If $ord(R) = n$ and $r_n = 1$, $R$ is called a monic operator. Let $E^{(n)}$ be the set of all elements in $\Psi DO$ of order at most $n$. Then, we have formally a direct sum decomposition

$$\Psi DO = DO \oplus E^{(-1)}$$

where $DO$ is the set of all differential operators ($R = R_+$ if $R \in DO$ $[R_1, R_2] = R_1 R_2 - R_2 R_1$).

We choose a subset of $\Psi DO$ whose elements $\Phi$ can be expressed by:

$$\Phi = 1 + u_{-1} \partial^{-1} + u_{-2} \partial^{-2} + \cdots.$$

The operators belonging to this set are invertible and forms a group, denoted by $(\Psi DO)^{\star}$.

We call a first order monic operator of the form:

$$L = \partial + u_{-1}\partial^{-1} + u_{-2}\partial^{-2} + \cdots$$

a Lax operator ($u_{-n} = u_{-n}(x,t)$). We set $L^n = L^{n-1} \cdot L$. Then,

$$L^n = \partial^n + q_{n-2}\partial^{n-2} + q_{n-3}\partial^{n-3} + \cdots.$$

We recall that if $R = \sum r_i \partial^i \in \Psi DO$ , then, $\frac{\partial R}{\partial t} = \sum \frac{\partial r_i}{\partial t}\partial^i$.

Let $L$ be a Lax operator. An equation of Lax type with $m$ brackets, formed from $L$ is an equation of the form:

$$\frac{\partial L}{\partial t} = \underbrace{[L, [L, \cdots [L, P]\cdots]]}_{m-times} \tag{5}$$

where $P \in DO$. In the rest of this paper we use the following notation of Mulase:

$$R \cdot [,Q]^m = \underbrace{[R, [R, \cdots [R, Q]\cdots]]}_{m-times}$$

Namely, $R \cdot [,Q] = [R, Q]$, $R \cdot [,Q]^2 = [R, [R, Q]]$ etc.

The possible operators $P$ in (5) must satisfy the requirement that $L \cdot [,P]^m \in E^{(-1)}$. Hence, such a pair $(P, L)$, where $P \in DO$ and $L$ is a Lax operator will be called *admissible*. If $L$ is a Lax operator, then, as in the classical case, for $n \in \mathbb{N}$, $\big((L^n)_+ , L\big)$ is an admissible pair.

**Lemma 1.** *Let $L$ be a Lax operator, $P \in DO$ such that $[L, P] \in E^{(-1)}$. Then, $P$ is a linear combination of the $(L^n)_+$, $n = 0, 1, \ldots$, with coefficients in $\mathbb{C}$ and $(P, L)$ is an admissible pair.*

*Proof.* This Lemma follows directly from the Lemma 3.1 of [9].

It should be noted that the condition $[L, P] \in E^{(-1)}$ for $P \in DO$ implies that the highest order coefficient of $P$ is a constant. On the other hand, if $(P, L)$ is an admissible pair, then, the highest order coefficient of $P$ is a polynomial of the form $(a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1x + a_0)$, where the $a_i$ are complex numbers for $i = 0, 1, \cdots m - 1$.

**Lemma 2.** *Let $L$ be a Lax operator, and $(P, L)$ be an admissible pair such that the top order coefficient $(a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1x + a_0)$ of $P$ is not constant, then, if $\operatorname{ord}(P) = n$, $P$ can be written in the following form*

$$P = (a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1x + a_0)(L^n)_+ + \sum_{k=0}^{n-1} r_k(L^k)_+ \tag{6}$$

*where $r_k \in \mathbb{C}[[x]]$ for every $k$.*

*Proof.* Suppose that the Lemma is true for all admissible $P \in DO$ of order less than $n$, such that the highest order coefficient is not constant. Let us take an admissible pair $(P, L)$, such that, $ord(P) = n$ and with the same property. Let $(a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1 x + a_0)$ be the highest order coefficient of $P$, then, we define:

$$L^n_{(a_{m-1}, \cdots, a_0)} = (a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1 x + a_0)(L^n)_+ + \sum_{k=0}^{n-1} q_k (L^k)_+$$

where the coefficients $q_k$ are formal power series that are obtained by imposing the condition that $((P - L^n_{(a_{m-1}, \cdots, a_0)}), L)$ is an admissible pair. To eliminate the coefficient of $\partial^{n-1}$ in $L \cdot [, (P - L^n_{(a_{m-1}, \cdots, a_0)})]^m$ we can take $q_{n-1} = p_{n-1} + (b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \cdots + b_1 x + b_0)$, where $p_{n-1}$ is the coefficient of $\partial^{n-1}$ in $P$ (because $\partial \cdot [, (b_{m-1}x^{m-1} + \cdots + b_1 x + b_0)]^m = 0$). Notice that we may assume that $b_{m-1} \neq 0$ or $b_{m-2} \neq 0$. Now, the equation equivalent to the annihilation of $\partial^k$ for $k = 0, 1, \ldots, n-2$ in $L \cdot [, (P - L^n_{(a_{m-1}, \ldots, a_0)})]^m$ only contains $q_k, q_{k+1}, \ldots, q_{n-1}$ and their derivatives, and it is of the form $q_k^{(m)} = Q_k(q_{k+1}, \ldots, q_{n-1})$, where $Q_k$ is a differential polynomial in $q_{k+1}, \ldots, q_{n-1}$ with coefficients in $\mathbb{C}[[x]]$. This fact allows us to calculate $q_{n-2}, q_{n-1}, \cdots, q_0$ recursively.

Since, $ord(P - L^n_{(a_{m-1}, \ldots, a_0)}) < n$ and the top order coefficient of $P - L^n_{(a_{m-1}, \ldots, a_0)}$ is not constant, we have:

$$P - L^n_{(a_{m-1}, \ldots, a_0)} = (b_{m-1}x^{m-1} + \cdots + b_1 x + b_0)(L^{n-1})_+ + \sum_{k=0}^{n-2} t_k (L^k)_+$$

where $t_k \in \mathbb{C}[[x]]$. Therefore, $P$ can be represented as a $\mathbb{C}[[x]]$-linear combination of $(L^k)_+$'s, which coincides with (6).

**Definition 1.** *Let $L$ be a Lax operator, then, the following system of infinitely many equations of Lax type with $m$ brackets is called the $m$-brackets hierarchy*

$$\frac{\partial L}{\partial t_n} = L \cdot \left[, (L^n)_+\right]^m. \tag{7}$$

We assume that the coefficients of $L$ in (7) are functions dependent on some additional variables $t_1, t_2, t_3, \ldots, t_n, \ldots$. We remark that $L = \partial$ is a trivial solution of (7).

Note that equation (7) can be interpreted as the compatibility condition for the following system of equations:

$$\begin{aligned} L\Phi &= \Phi\partial, \\ \frac{\partial \Phi}{\partial t_n} &= L \cdot \left[, (L^n)_-\right]^{m-1} \Phi. \end{aligned}$$

for a formal $(\Psi DO)^\star$-function $\Phi(x, t_1, t_2, \ldots)$.

In many cases it is possible and convenient to represent the Lax operator $L$ in a *formal dressing* form $L = \Phi \partial \Phi^{-1}$, where $\Phi \in (\Psi DO)^\star$ is determined up to multiplication on the right by a series in $\partial^{-1}$ with constant coefficients (independents of $x$, but can be depend about the time variables $t_1, t_2, \ldots$) starting with 1. In terms of $\Phi$, the equations of the $m$-brackets hierarchy are:

$$\frac{\partial \Phi}{\partial t_n} = L \cdot \left[, (L^n)_- \right]^{m-1} \Phi. \tag{8}$$

In fact, let $\Phi$ be a solution of (8), where $L = \Phi \partial \Phi^{-1}$ then $L$ is a Lax operator that satisfies (7). To see this, notice that from (8) we immediately obtain the equation

$$\frac{\partial \Phi^{-1}}{\partial t_n} = -\Phi^{-1} \left( L \cdot \left[, (L^n)_- \right]^{m-1} \right). \tag{9}$$

Now,

$$\frac{\partial L}{\partial t_n} = \frac{\partial \Phi}{\partial t_n} \partial \Phi^{-1} + \Phi \partial \frac{\partial \Phi^{-1}}{\partial t_n}. \tag{10}$$

If we replace $\frac{\partial \Phi}{\partial t_n}$ and $\frac{\partial \Phi^{-1}}{\partial t_n}$ in (10) by the right hand side of (8) and (9) respectively, we have:

$$\begin{aligned}
\frac{\partial L}{\partial t_n} &= \left( L \cdot \left[, (L^n)_- \right]^{m-1} \right) \Phi \partial \Phi^{-1} - \Phi \partial \Phi^{-1} \left( L \cdot \left[, (L^n)_- \right]^{m-1} \right) \\
&= -L \cdot \left[, (L^n)_- \right]^{m} \\
&= L \cdot \left[, (L^n)_+ \right]^{m}.
\end{aligned}$$

Now we can prove the following lemma:

**Lemma 3.** *Let* $L = \Phi \partial \Phi^{-1}$, *where* $\Phi \in (\Psi DO)^\star$, *then,* $(n, m \geq 1)$

$$L \cdot \left[, L_-^n \right]^{m-1} \Phi = \Phi \left( \partial \cdot \left[, \Phi^{-1} L_-^n \Phi \right]^{m-1} \right). \tag{11}$$

*Proof.* We will prove it by induction. First we prove the formula:

$$L \cdot \left[, L_-^n \right] \Phi = \Phi \left( \partial \cdot \left[, \Phi^{-1} L_-^n \Phi \right] \right).$$

In fact,

$$\begin{aligned}
L \cdot \left[, L_-^n \right] \Phi &= \left[ L, L_-^n \right] \Phi \\
&= \left( \Phi \partial \Phi^{-1} L_-^n - L_-^n \Phi \partial \Phi^{-1} \right) \Phi \\
&= \Phi \left( \partial \Phi^{-1} L_-^n \Phi - \Phi^{-1} L_-^n \Phi \partial \right) \\
&= \Phi \left[ \partial, \Phi^{-1} L_-^n \Phi \right] \\
&= \Phi \left( \partial \cdot \left[, \Phi^{-1} L_-^n \Phi \right] \right).
\end{aligned}$$

Thus, the equality (11) holds. Let (11) be true for $m = k$, then,

$$
\begin{aligned}
\left(L \cdot [, L_-^n]^{(k+1)-1}\right) \Phi &= \left\{ L \left(L \cdot [, L_-^n]^{k-1}\right) - \left(L \cdot [, L_-^n]^{k-1}\right) L \right\} \Phi \\
&= \left\{ \Phi \partial \Phi^{-1} \left(\left(L \cdot [, L_-^n]^{k-1}\right) \Phi\right) \right. \\
&\quad \left. - \left(\left(L \cdot [, L_-^n]^{k-1}\right) \Phi\right) \partial \right\} \\
&= \Phi \partial \Phi^{-1} \left(\Phi \left(\partial \cdot [, \Phi^{-1} L_-^n \Phi]^{k-1}\right)\right) \\
&\quad - \Phi \left(\partial \cdot [, \Phi^{-1} L_-^n \Phi]^{k-1}\right) \partial \\
&= \Phi \left\{ \partial \left(\partial \cdot [, \Phi^{-1} L_-^n \Phi]^{k-1}\right) \right. \\
&\quad \left. - \left(\partial \cdot [, \Phi^{-1} L_-^n \Phi]^{k-1}\right) \partial \right\} \\
&= \Phi \left(\partial \cdot [, \Phi^{-1} L_-^n \Phi]^{k}\right).
\end{aligned}
$$

**Lemma 4.** *The operator*

$$
Z^c = \partial \cdot \left[, - \left(H_0^c + \Phi_0^{-1} \frac{\partial \Phi_0}{\partial t_n}\right)\right]^{m-1} - \Phi_0^{-1} \frac{\partial \Phi_0}{\partial t_n}
$$

*where*

$$
H_0^c = \Phi_0^{-1} \left(-L_-^n\right) \Phi_0 - \Phi_0^{-1} \frac{\partial \Phi_0}{\partial t_n} \quad \text{and } L = \Phi_0^{-1} \partial \Phi_0
$$

*has only constant coefficients.*

*Proof.* Lemma 4 shows that

$$
\begin{aligned}
[\partial, Z^c] &= \left[\partial, \partial \cdot [, \Phi_0^{-1} \left(L_-^n\right) \Phi_0]^{m-1} - \Phi_0^{-1} \frac{\partial \Phi_0}{\partial t_n}\right] \\
&= \left[\partial, \Phi_0^{-1} \left(L \cdot [, \left(L_-^n\right)]^{m-1}\right) \Phi_0 - \Phi_0^{-1} \frac{\partial \Phi_0}{\partial t_n}\right] \\
&= -\Phi_0^{-1} \left(L \cdot [, - \left(L_-^n\right)]^{m}\right) \Phi_0 - \Phi_0^{-1} \left[L, \frac{\partial \Phi_0}{\partial t_n} \Phi_0^{-1}\right] \Phi_0 \\
&= \Phi_0^{-1} \left(- \left(L \cdot [, \left(L_+^n\right)]^{m}\right) + \left[\frac{\partial \Phi_0}{\partial t_n} \Phi_0^{-1}, L\right]\right) \Phi_0 \\
&= \Phi_0^{-1} \left(-\frac{\partial L}{\partial t_n} + \frac{\partial L}{\partial t_n}\right) \Phi_0 \\
&= 0.
\end{aligned}
$$

**Theorem 1.** *Let $L$ be a Lax operator such that $L$ satisfies equation (7) for $n$ fixed, then, there is an operator $\Phi \in (\Psi DO)^\star$, where $L = \Phi \partial \Phi^{-1}$, satisfying the equation (8) for this $n$.*

*Proof.* Suppose $L$ satisfies the equation (7). Let $\Phi_0 \in ()^\star$ be an operator such that $L = \Phi_0 \partial \Phi_0^{-1}$, then, from the Lemma 5, there is a constant element $C \in (\Psi DO)^\star$ (See e.g. [12]) such that

$$\partial \cdot \left[, -\left(H_0^c + \Phi_0^{-1}\frac{\partial \Phi_0}{\partial t_n}\right)\right]^{m-1} - \Phi_0^{-1}\frac{\partial \Phi_0}{\partial t_n} = \frac{\partial C}{\partial t_n}C^{-1}.$$

Hence

$$(\Phi_0 C)^{-1}\left(L \cdot \left[, (L_-^n)\right]^{m-1} - \frac{\partial(\Phi_0 C)}{\partial t_n}(\Phi_0 C)^{-1}\right)(\Phi_0 C)$$

$$= C^{-1}\left(\partial \cdot \left[, -\left(H_0^c + \Phi_0^{-1}\frac{\partial \Phi_0}{\partial t_n}\right)\right]^{m-1} - \Phi_0^{-1}\frac{\partial \Phi_0}{\partial t_n} - \frac{\partial C}{\partial t_n}C^{-1}\right)C$$

$$= 0.$$

Let $\Phi = \Phi_0 C$. Note that $\Phi \partial \Phi^{-1} = (\Phi_0 C)\partial(\Phi_0 C)^{-1} = \Phi_0 \partial \Phi_0^{-1} = L$. Thus, we have found an operator $\Phi \in (\Psi DO)^\star$ satisfying equation (8). $\blacksquare$

## 3   Invariant Polynomial Functions

In this section we will work with pseudodifferential operators (4). A conservation law is an identity

$$\frac{\partial H}{\partial t} = \frac{\partial J}{\partial x}$$

that follows formally from (5). The *conserved density* $H$ and *flux* $J$ are differential polynomials in $u_{-1}, u_{-2}, u_{-3}, \ldots$ and their $x$-derivatives $u_{-n}^{(k)}$. An invariant polynomial functional for (5) is a function of the form:

$$F(H) = \int_0^1 H\,dx$$

where $H$ is a conserved density. Notice that if $F$ is an invariant polynomial function, then, $\frac{\partial F}{\partial t} = 0$.

For $R \in \Psi DO$ we define the residue as:

$$res\,R = r_{-1}.$$

Next, we shall also use the Adler function

$$Tr\,R = \int_0^1 res\,R\,dx.$$

This functional has the property $Tr\,[R_1, R_2] = 0$, for every $R_1, R_2 \in \Psi DO$ (see the proof of Theorem 8).

**Lemma 5.** *For any $k \geq 2$, by virtue of the equations (7), then,*

$$\frac{\partial L^k}{\partial t_n} = \left[L^k, \left(L \cdot \left[, (L^n)_+\right]^{m-1}\right)\right]$$

*holds.*

*Proof.* Let $k = 2$, then,

$$
\begin{aligned}
\frac{\partial L^2}{\partial t_n} &= \frac{\partial L}{\partial t_n}L + L\frac{\partial L}{\partial t_n} \\
&= \left(L \cdot [,(L^n)_+]^m\right)L + L\left(L \cdot [,(L^n)_+]^m\right) \\
&= \left(L\left(L \cdot [,(L^n)_+]^{m-1}\right) - \left(L \cdot [,(L^n)_+]^{m-1}\right)L\right)L \\
&\quad + L\left(L\left(L \cdot [,(L^n)_+]^{m-1}\right) - \left(L \cdot [,(L^n)_+]^{m-1}\right)L\right) \\
&= L^2\left(L \cdot [,(L^n)_+]^{m-1}\right) - \left(L \cdot [,(L^n)_+]^{m-1}\right)L^2 \\
&= \left[L^2, \left(L \cdot [,(L^n)_+]^{m-1}\right)\right].
\end{aligned}
$$

Now, we suppose that the lemma is true for $k = r$, i.e.

$$
\frac{\partial L^r}{\partial t_n} = \left[L^r, \left(L \cdot [,(L^n)_+]^{m-1}\right)\right].
$$

We have:

$$
\begin{aligned}
\frac{\partial L^{r+1}}{\partial t_n} &= \frac{\partial L^r}{\partial t_n}L + L^r\frac{\partial L}{\partial t_n} \\
&= \left[L^r, \left(L \cdot [,(L^n)_+]^{m-1}\right)\right]L + L^r\left(L \cdot [,(L^n)_+]^m\right) \\
&= \left(L^r\left(L \cdot [,(L^n)_+]^{m-1}\right) - \left(L \cdot [,(L^n)_+]^{m-1}\right)L^r\right)L \\
&\quad + L^r\left(L\left(L \cdot [,(L^n)_+]^{m-1}\right) - \left(L \cdot [,(L^n)_+]^{m-1}\right)L\right) \\
&= L^{r+1}\left(L \cdot [,(L^n)_+]^{m-1}\right) - \left(L \cdot [,(L^n)_+]^{m-1}\right)L^{r+1} \\
&= \left[L^{r+1}, \left(L \cdot [,(L^n)_+]^{m-1}\right)\right].
\end{aligned}
$$

**Theorem 2.** *The polynomial functionals*

$$
F_k = Tr\, L^k = \int_0^1 res\, L^k\, dx \qquad k = 1, 2, 3, \cdots
$$

*are invariant polynomial functionals.*

*Proof.* It is well know that $res\,[R_1, R_2] = \partial h$ (see e.g. [5]), where $h$ is a differential polynomial in the coefficients of $R_1$ and $R_2$, $R_1, R_2 \in \Psi DO$. Hence

$$
\begin{aligned}
\frac{\partial}{\partial t_n}(res\, L^k) &= res\,\frac{\partial L^k}{\partial t_n} = res\,\left[L^k, \left(L \cdot [,(L^n)_+]^{m-1}\right)\right] \\
&= \frac{\partial J_k}{\partial x}
\end{aligned}
$$

where $J_k$ are differential polynomials in $u_{-n}$, $n = 1, 2, 3, \ldots$; and their $x$-derivatives $u_{-n}^{(i)}$, $i, n = 1, 2, 3, \ldots$.

## 4   Zakharov - Shabat Type Equations and Some New Integral Systems

The purpose of this section is to show how the system (7) generates new zero-curvature type equations.

**Theorem 3.** *Let $B_q = (L^q)_+$, $q \geq 1$. Equations (7) imply that:*

$$\frac{\partial B_n}{\partial t_s} - \frac{\partial B_s}{\partial t_n} = \left( L \cdot [, [B_n, B_s]]^{m-1} \right)_+ . \tag{12}$$

*Further, every equation given by (12) is equivalent to a finite system of equations, the number of equations being equal to the number of unknown quantities, i.e. is equivalent to a closed system of partial differential equations.*

*Proof.* It is easy to see that if $q \geq 1$, then,

$$\left[ L^q, \left( L \cdot [, (L^n)_+]^{m-1} \right) \right] = L \cdot [, [L^q, (L^n)_+]]^{m-1} . \tag{13}$$

In fact, for $B \in \Psi DO$ we have:

$$L^q \cdot [, [L, B]] = L \cdot [, [L^q, B]] ,$$

(See [6] for more details). Now, this result implies (13).

¿From (13), and according to lemma 7,

$$
\begin{aligned}
\frac{\partial B_n}{\partial t_s} - \frac{\partial B_s}{\partial t_n} &= \left( L \cdot [, [L^n, (L^s)_+]]^{m-1} \right)_+ - \left( L \cdot [, [L^s, (L^n)_+]]^{m-1} \right)_+ \\
&= \left( L \cdot [, [L^n, B_s] - [L^s, B_n]]^{m-1} \right)_+ \\
&= \left( L \cdot [, [B_n, B_s]]^{m-1} \right)_+ - \left( L \cdot [, [(L^n)_- , (L^s)_-]]^{m-1} \right)_+ \\
&= \left( L \cdot [, [B_n, B_s]]^{m-1} \right)_+ .
\end{aligned}
$$

## 5   The Baker - Akhiezer Function

The main purpose of this section is to prove a bilinear identity with several brackets for the Baker-Akhiezer and the ad joint Baker-Akhiezer functions (the proof of the bilinear identities is essentially the same as in [6]); they will play a crucial role in obtaining "formal solutions" of the hierarchy (7).

A formal eigen function $W$ of $L$ with eigenvalue $z$, the so-called Baker-Akhiezer function, is very important in this theory (See [4]). For a solution

$$L = \partial + R_{-1}\partial^{-1} + \cdots$$

to the $m-$bracket hierarchy (8), we have, in terms of the dressing operator $\Phi \in (\Psi DO)^*$, the associated Baker-Akhiezer function:

$$W = \Phi \exp \xi \left( x, \widetilde{t}, z \right)$$

(Recall that $L = \Phi \partial \Phi^{-1}$) where $\xi\left(x, \widetilde{t}, z\right) = xz + \left(\sum_{k=1}^{\infty} t_k z^k\right)$, satisfying the equations:

$$LW = zW, \quad \frac{\partial W}{\partial t_n} = \left(L \cdot [, L_-^n]^{m-1}\right) W + z^n W.$$

The second equation yields:

$$\tau_n W = B_n W \tag{14}$$

where $\tau_n = \frac{\partial}{\partial t_n} - \left(L \cdot [, L_-^n]^{m-1}\right) - L_-^n$. Also, the equation (18) can be written in the following form:

$$\tau_n \Phi = -L_-^n \Phi \tag{15}$$

The ad joint Baker-Akhiezer function is:

$$W^a = (\Phi^*)^{-1} \exp\left(-\xi\left(x, \widetilde{t}, z\right)\right)$$

where the star means the conjugation, i.e. if $X = \sum p_i \partial^i$, then, $X^* = \sum (-\partial)^i p_i^*$ where $p_i^*$ is the transpose of $p_i$.

**Proposition 1.** *Suppose we have a solution $L$ of (7) and $L = \Phi \partial \Phi^{-1}$. Then, the following "double bracket bilinear identity"*

$$res_z \left(\partial^i \tau_n^j W\right) \cdot (W^a) = 0$$

*holds for any $(i, j, n)$ and $i \geq 0$, $j = 0, 1$ and $n \geq 1$, where $\tau_n^0 = id$. Here $W$ and $W^a$ are the Baker-Akhiezer and ad joint Baker-Akhiezer function respectively.*

The notation $res_z$ means the coefficient of $z^{-1}$. The notation $res_\partial$ means the coefficient of $\partial^{-1}$.

*Proof.* Since $\tau_n W = B_n W$, it is sufficient to prove this equality for $j = 0$ and $i \geq 0$, but in this case the proof is well known (see e.g. [5]) and therefore, it is omitted.

It is worth mentioning that if there are two expressions

$$\psi = \sum_{i=0}^{\infty} \psi_i\left(x, \widetilde{t}, z\right) z^{-i} e^{\xi\left(x, \widetilde{t}, z\right)}, \quad \psi^a = \sum_{i=0}^{\infty} \psi_i^a\left(x, \widetilde{t}, z\right) z^{-i} e^{-\xi\left(x, \widetilde{t}, z\right)} \tag{16}$$

with $\psi_0 = 1$, $\psi_0^a = 1$, such that

$$res_z \left(\partial^i \psi\right) \cdot (\psi^a) = 0, \ i = 0, 1, 2, \cdots$$

then, letting $\Phi = \sum_{i=0}^{\infty} \psi_i \partial^{-i}$ we will have $\psi = W = \Phi \exp \xi\left(x, \widetilde{t}, z\right)$ and $\psi^a = W^a = (\Phi^*)^{-1} \exp\left(-\xi\left(x, \widetilde{t}, z\right)\right)$.

Now, we formulate the converse of proposition 10.

**Proposition 2.** *If for the expressions (20) with $\psi_0 = 1$, $\psi_0^a = 1$ we have*

$$res_z \left( \partial^i \tau_n^j \psi \right) \cdot (\psi^a) = 0, \; i, j = 0, 1, 2, \cdots$$

*Then, an operator $L = \partial + R_{-1} \partial^{-1} + \cdots$ can be constructed such that $L$ satisfies (7).*

*Proof.* Let $\Phi = \sum_{i=0}^{\infty} \psi_i \partial^{-i}$ and $L = \Phi \partial \Phi^{-1}$, then,

$$
\begin{aligned}
\left( \tau_n \Phi + L_-^n \Phi \right) e^{\xi(\widetilde{t}, z)} &= \left( \frac{\partial \Phi}{\partial t_n} - \left( L \cdot [, L_-^n]^{m-1} \right) \Phi \right) e^{\xi(x, \widetilde{t}, z)} \\
&= \left( \frac{\partial \Phi}{\partial t_n} - \left( L \cdot [, L_-^n]^{m-1} \right) \Phi \right) e^{\xi(x, \widetilde{t}, z)} \\
&\quad + \Phi \frac{\partial e^{\xi(x, \widetilde{t}, z)}}{\partial t_n} - \Phi \frac{\partial e^{\xi(x, \widetilde{t}, z)}}{\partial t_n} \\
&= \left( \frac{\partial \Phi e^{\xi(x, \widetilde{t}, z)}}{\partial t_n} - \Phi z^n e^{\xi(x, \widetilde{t}, z)} \right. \\
&\quad \left. - \left( L \cdot [, L_-^n]^{m-1} \right) \Phi e^{\xi(x, \widetilde{t}, z)} \right) \\
&= \left( \frac{\partial \Phi e^{\xi(x, \widetilde{t}, z)}}{\partial t_n} - \Phi \partial^n e^{\xi(x, \widetilde{t}, z)} \right. \\
&\quad \left. - \left( L \cdot [, L_-^n]^{m-1} \right) \Phi e^{\xi(x, \widetilde{t}, z)} \right) \\
&= \left( \frac{\partial}{\partial t_n} - L^n - \left( L \cdot [, L_-^n]^{m-1} \right) \right) \Phi e^{\xi(x, \widetilde{t}, z)} \\
&= \left( \tau_n - B_n \right) \Phi e^{\xi(x, \widetilde{t}, z)}.
\end{aligned}
$$

Now, according to the assumption

$$
\begin{aligned}
0 &= res_z \left( \partial^i \left( \tau_n - B_n \right) \Phi e^{\xi(x, \widetilde{t}, z)} \cdot \left( (\Phi^*)^{-1} e^{-\xi(x, \widetilde{t}, z)} \right) \right) \\
&= res_z \left( \partial^i \left( \tau_n \Phi + L_-^n \Phi \right) e^{\xi(x, \widetilde{t}, z)} \cdot \left( (\Phi^*)^{-1} e^{-\xi(x, \widetilde{t}, z)} \right) \right) \\
&= res_\partial \left( \partial^i \left( \tau_n \Phi + L_-^n \Phi \right) \cdot \Phi^{-1} \right)
\end{aligned}
$$

for $i = 0, 1, 2, \cdots$, and this gives that

$$\left( \tau_n \Phi + L_-^n \Phi \right) \cdot \Phi^{-1} = 0.$$

Hence,

$$\left( \tau_n \Phi + L_-^n \Phi \right) = 0.$$

## 6    Open Problems

A problem for further work is to find solutions of the hierarchies using the Tau functions (see e.g. [5]) which must exist according to the results obtained. The

problem is to find what forms the Tau functions can have. For example, for the equations of the Brockett hierarchy one step to follow is to study the Tau function for the first equation of the hierarchy; i.e., find its expression in Hirota form and then calculate the explicit form of the Tau function. Furthermore, one should be able to show that similar results exist for all the hierarchy. For the existence of the Tau function, we can assume that all these hierarchies are constituted by commutative laws. In this paper we have not shown this commutability. However, good results can be obtained by showing commutability in the general case.

A very interesting problem arises here, namely to try to extend the approach of Mulase for the hierarchy $KP$ to the hierarchies with several brackets; we do not know what results of factorization for pseudo differentials operators are lacking. In order to attack this problem one has to start with the study of the Brockett hierarchy, and in this case it consists of linearizing the hierarchy using theorems of factorization of the Birkhoff type for groups of Loops. This problem apparently is very difficult.

Another topic of investigation that could be followed is to study discrete hierarchies of equations of Lax type with several brackets using the shift matrix. This subject must have connection, as in the case of the discreet $KP$, with matrices of moments and orthogonal polynomials. Here, one can also consider to extend the method of Mulase in this context, as done by Felipe and Ongay (see [8]) for the discrete $KP$.

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