A GROWTH FUNCTIONAL EQUATION

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Abstract

We state a growth functional equation related with the Von Bertalanffy physiological differential equation

$$\frac{dw}{dt} = nw^d - kw^m$$

This functional is useful in computing solutions of the latter.

1. Introduction.

By studying the animal physiology *Von Bertalanffy* stated that the increase in weight w of an animal is due to quantitative differences between the biological processes of anabolism and catabolism. This can be expressed by the differential equation.

$$\frac{dw}{dt} = nw^d - kw^m, \quad w > 0 \tag{1}$$

where n, k, d and m are real parameters, and n and k correspond to the rates of anabolism and catabolism respectively. (See [FAB], [SPA]).

Due to the importance of equation [1] in the study of animal growth, we want to present here a method which allows to simplify the problem of its integration. We state a functional equation which is very useful in computing solutions of equation [1].

2. A Functional Equation.

We can see that in particular cases equation [1] can be integrated easily. For example, if $d = 2/3$ and $m = 1$ we obtain the equation

$$\frac{dw}{dt} = nw^{2/3} - kw, \quad w > 0 \qquad [2]$$

which together with the isometrical growth assumptions $w = \ell^3$ and $w(t_0) = 0$ becomes

$$\begin{cases} \dfrac{d\ell}{dt} & = \dfrac{n}{3} - \dfrac{k}{3}\ell \\ \ell(t_0) & = 0 \end{cases} \qquad [3]$$

The latter has solution $\ell = \frac{n}{k}\left(1 - e^{-\frac{k}{3}(t-t_0)}\right)$, obtained easily by separation of variables.

Even though in general we will not find an explicit expression for w, let us apply separation of variables to [1]. We still use the assumption $w(t_0) = 0$. We get

$$t_w - t_0 = \int_0^w \frac{d\alpha}{n\alpha^d - k\alpha^m}, \quad w > 0.$$

Now, if we make the change of variable $w = \ell^s, s > 0$, we obtain the functional equation.

$$t(d, m, w) = s\ t(sd - s + 1, sm - s + 1, w^{1/s}) \qquad [4]$$

where

$$t(d, m, w) = \textstyle\int_0^w \dfrac{1}{(n\alpha^d - k\alpha^m)} d\alpha \qquad [5]$$

3. Applications.

In the case of isometrical growth we have seen at the begining of § 2 that

$$t(2/3, 1, w) = -\frac{3}{k}\ln\left[1 - \frac{k}{n}w^{1/3}\right] \qquad [6]$$

By using this particular result and equation [4] we can easily compute $t(d, 1, w)$ with $d \neq 1$, i.e. when $m = 1$. In fact by [4].

$$t(d, 1, w) = s\ t(sd - s + 1, 1, w^{1/s}), \qquad [7]$$

and if we put $s = 1/3(1 - d)$ and use [6] we get

$$\begin{aligned} t(d,1,w) &= \frac{1}{3(1-d)}t\left(\frac{2}{3},1,w^{3(1-d)}\right) \\ &= -\frac{1}{k(1-d)}\ln\left(1-\frac{k}{n}w^{1-d}\right) \end{aligned}$$

Observe that $t(d,m,w)$ has the same expression as $-t(m,d,w)$ but interchanging n and k. Then we have that

$$\begin{aligned} t(1,m,w) &= -\frac{1}{n(1-m)}\ln\left[1-\frac{n}{k}w^{1-m}\right] \\ &= \int_0^w \frac{d\alpha}{n\alpha-k\alpha^m}. \end{aligned}$$

if $m \neq 1$. A particular case of this is when $m=2$, which gives us the solution of the logistic equation:

$$t(1,2,w) = -\frac{1}{n}\ln\left(1-\frac{n}{k}w^{-1}\right).$$

4. Convergence of the Integral $t(d,m,w)$

Because of § 3 it is enough to study the convergence of $t(d,m,w)$ when $d \neq 1$ and $m \neq 1$. Let $s = 1/(m-1)$ in [4]. Then we get

$$t(d,m,w) = \frac{1}{m-1}t\left(\frac{d+m-2}{m-1},2,w^{m-1}\right) \qquad [8]$$

We note the integral in the right hand side of [8] $J_r(m,w)$, where $r = \dfrac{d+m-2}{m-1}$, i.e.

$$J_r(m,w) = \int_0^{w^{m-1}} \frac{d\alpha}{n\alpha^r - k\alpha^2} = \int_{w^{1-m}}^{\infty} \frac{dz}{nz^{2-r}-k}.$$

This improper integral converges if $r<1$ and $w^{1-m} > (k/n)^{1/(2-r)}$. (See [HAR, 1967, p. 359]).

The condition $r = \dfrac{d+m-2}{m-1} < 1$ implies that $m>1$ and $d<1$.

Therefore the functional equation [8] has for its domain the region

$$R = \left\{(d,m,w) : d<1,\ m>1,\ w<\left(\frac{n}{k}\right)^{\frac{1}{(2-r)(m-1)}}\right\}$$

A similar developing is obtained if we take $s = \frac{1}{d-1}$ in [4]. In this case the functional equation:

$$t(d,m,w) = \frac{1}{d-1} t\left(2,\ \frac{m+d-2}{d-1},\ w^{d-1}\right)$$

has the following set as domain:

$$R = \left\{(d,m,w) : d > 1,\ m < 1,\ w < \left(\frac{k}{n}\right)^{\frac{1}{(2-r)(d-1)}}\right\}$$

For example if we are dealing with the equation

$$\frac{dw}{dt} = nw^{0.8} - kw^{1.7},$$

the associated integral is given by

$$\begin{aligned} t(0.8, 1.7, w) &= \frac{10}{7} J_{5/7}(1.7,\ w) \\ &= \frac{10}{7} \int_{w^{-0.7}}^{\infty} \frac{dz}{nz^{9/7} - k} \end{aligned}$$

The latter converges for $w < \left(\frac{n}{k}\right)^{\frac{10}{9}}$.

5. References.

[FAB] Fabens, Augustus J. *Properties and Fitting of the Von Bertalanffy Growth Curve.* Growth, 1965, 29, 265-289.

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