

RIGOROUS PROOFS FOR CONE-BEAM RECONSTRUCTION

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Abstract

Cone-beam reconstruction is a three-dimensional tomographic reconstruction technique with the potential capability of producing images with temporal contrast, and three-dimensional spatial resolution.

In this paper we present a rigorous proof of the novel three-dimensional inversion formula developed by B. Smith and derive strict upper bounds for the error.

1. Introduction

Cone-Beam reconstruction is a three-dimensional tomographic technique with the potential capability of producing images with improved temporal and three-dimensional spatial resolution. In the past, very few inversion formulas were known and it was not clear what type of data collection geometries would produce enough information so that a “practical” inversion

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formula would be possible. See Marr [7], Louis and Natterer [5], and the references therein. Lately, however, several reconstruction methods and data collection geometries, which will significantly increase the attractiveness of implementing cone-beam tomography in both medical and non medical applications, have been derived formally. See Feldkamp et al [1], Finch [2] and Grangeat [3].

In this paper we concentrate our attention on one of such new formulas initially proposed by B.K.P. Horn [4] for arbitrary two-dimensional raysampling schemes and more recently extended to three dimensions by B. Smith [10], [11], and attempt to enhance its mathematical foundation. We hope that making the formal derivations rigorous will increase the acceptance of this work by other researchers, and at the same time, lay a solid background for further advances in the present theory of cone-beam reconstructions as well as in the analysis of many numerical and computational related questions.

The paper is organized as follows: In Section 2, the mathematical foundation of the new three-dimensional Radon oinversion formula is investigated assuming the existence (and availability) of a complete set of projection data. Recognizing that the reconstruction by computerized tomography procedure is inherently an ill-posed problem in the sense that small errors in the projection data might cause large errors in the computed density function- it becomes necessary to stabilize the inverse problem. This task is addressed in Section 3 where we also present new rigorous upper bounds for the error as a function of the regularizing parameters and a theoretical proof of convergence.

2. Mathematical Foundation of the New Three-dimensional Radon Inversion Formula

In this section we shall consider the three-dimensional Radon Transform $\tilde{f} = \mathcal{R}f$ of density functions $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, defined by

$$\tilde{f}(l, \beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(l\beta + s\beta_1 + t\beta_2) ds dt ,$$

where β , β_1 and β_2 form a set of orthonormal three-dimensional vectors. This formula corresponds to integration over the plane spanned by the vectors β_1 and β_2 , perpendicular to the β direction and passing through the point $l\beta$, $-\infty < l < \infty$. See Smith [11] for details.

It is well known (see Ludwig [6]), that the three-dimensional Radon Inver-

sion Formula is given by

$$f(x) = -\frac{1}{8\pi^2} \int_{\|\beta\|=1} \frac{\partial^2}{\partial l^2} \tilde{f}(l, \beta)|_{l=x \cdot \beta} d\beta . \tag{1}$$

We observe that the inversion formula corresponds to the integration of $\frac{\partial^2}{\partial l^2} \tilde{f}(l, \beta)$ over all the planes passing through a given reconstruction point.

Formula (1) also shows that the ill-posedness of the three-dimensional reconstruction procedure is related with the fact that the data \tilde{f} has to be partially differentiated twice. In the presence of noise in the data, the numerical differentiation algorithm must be regularized to restore stability with respect to the data. (See for example Murio [8]).

In order to establish a new inversion formula more suitable for numerical computation – the main result in this section – we shall require the continuity of $\frac{\partial^4 \tilde{f}}{\partial l^4}(l, \beta)$. Notice that the continuity of $\frac{\partial^2}{\partial t^2} \tilde{f}(l, \beta)$ is sufficient for the existence of the reconstructed density function $f(x)$ according to formula (1).

We also assume that all the functions involved, and their derivatives, decrease at infinity more rapidly than $\frac{1}{\|x\|}$.

In this setting we always consider the functions with compact support already extended (smoothly) to the entire space so that the previous assumptions are satisfied.

In what follows we shall also use the one-dimensional Fourier transform of a function $f(t)$, defined by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-it\omega} dt \quad , \quad -\infty < \omega < \infty \quad ,$$

and the corresponding inverse Fourier transform given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{-it\omega} d\omega \quad , \quad -\infty < t < \infty \quad .$$

The Hilbert Transform of a function $g : \mathbb{R} \rightarrow \mathbb{R}$ will play an interesting role in the sequel. This singular integral transform is defined by

$$(\mathcal{H}g)(s) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{1}{s-t} g(t) dt \quad .$$

The integral is understood in the sense of the Cauchy principal value and has meaning for functions $g(t)$, defined for $-\infty < t < \infty$ and tending to zero together with its continuous first derivative not more slowly than $\frac{1}{|t|}$ as $|t| \rightarrow \infty$.

Now, we are ready for the introduction of the important function

$$\begin{aligned} F(l, \beta) &\equiv \left(\mathcal{H} \frac{\partial \tilde{f}}{\partial t} \right) (l, \beta) \\ &\equiv \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{1}{l-t} \frac{\partial \tilde{f}}{\partial t} (t, \beta) dt, \end{aligned} \quad (2)$$

where the Hilbert transform is understood with respect to the first variable. The function $F(l, \beta)$ (see Smith [11]) is fundamental for the development of the new three-dimensional Radon inversion formula as well as for the consistency and stability of the method. To obtain a different expression for $F(l, \beta)$, we shall work in a slightly different way.

Lemma 1. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be such that $g'(t)$ and $g''(t)$ are continuous and vanish at infinity together with $g(t)$ faster than $\frac{1}{|t|}$; then

$$(\mathcal{H}g')(l) = \frac{1}{\pi} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} H_{\alpha}(l-t)g(t)dt,$$

where

$$H_{\alpha}(s) = \begin{cases} 1/\alpha^2, & |s| < \alpha, \\ -1/s^2, & |s| \geq \alpha. \end{cases} \quad (3)$$

Proof. Let

$$G(l) = (\mathcal{H}g')(l) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{1}{l-t} g'(t) dt. \quad (4)$$

After an integration by parts in equation (4), we have

$$G(l) = \frac{1}{\pi} \int_{-\infty}^{\infty} g''(t) \log |l-t| dt. \quad (5)$$

A different integration by parts in equation (4) produces

$$G(l) = \frac{1}{\pi} \lim_{\alpha \rightarrow 0} \left[\frac{1}{\alpha} (g(l - \alpha) + g(l + \alpha)) - \int_{-\infty}^{l-\alpha} \frac{1}{(l-t)^2} g(t) dt - \int_{l+\alpha}^{\infty} \frac{1}{(l-t)^2} g(t) dt \right]. \tag{6}$$

Using Taylor expansions we obtain

$$\frac{1}{\alpha} [g(l - \alpha) + g(l + \alpha)] = \frac{2}{\alpha} g(l) + \frac{1}{2} \alpha [g''(\xi_1) + g''(\xi_2)],$$

$$l - \alpha < \xi_1 < l < \xi_2 < l + \alpha,$$

and

$$\frac{1}{\alpha^2} \int_{l-\alpha}^{l+\alpha} g(t) dt = \frac{2}{\alpha} g(l) + \frac{\alpha}{6} [g''(\xi_3) + g''(\xi_4)],$$

$$l - \alpha < \xi_3 < l < \xi_4 < l + \alpha.$$

Thus,

$$\frac{1}{\alpha} [g(l - \alpha) + g(l + \alpha)] - \frac{1}{\alpha^2} \int_{l-\alpha}^{l+\alpha} g(t) dt = 0(\alpha), \tag{7}$$

where $0(\alpha) \leq \frac{4}{3} \alpha \|g''\|_{\infty}$ and

$$\frac{1}{\pi} \lim_{\alpha \rightarrow 0} \left[\frac{1}{\alpha} [g(l - \alpha) + g(l + \alpha)] - \frac{1}{\alpha^2} \int_{l-\alpha}^{l+\alpha} g(t) dt \right] = 0. \tag{8}$$

Subtracting equation (8) from equation (6) we get

$$G(l) = \frac{1}{\pi} \lim_{\alpha \rightarrow 0} \left[\frac{1}{\alpha^2} \int_{l-\alpha}^{l+\alpha} g(t) dt - \int_{-\infty}^{l-\alpha} \frac{1}{(l-t)^2} g(t) dt - \int_{l+\alpha}^{\infty} \frac{1}{(l-t)^2} g(t) dt \right],$$

and using the definition of the kernel H_{α} , given by equation (3), we complete the proof.

Remarks.

1. The kernel function H_{α} was introduced by Horn, (See Horn [4]) for general two-dimensional fan-beam reconstruction algorithms.

2. Applying Lemma 1, with $g(t) = \tilde{f}(t, \beta)$ and β fixed, we get

$$F(l, \beta) = \frac{1}{\pi} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} H_{\alpha}(l-t) \tilde{f}(t, \beta) dt . \quad (9)$$

Our next task is to show that the three dimensional Radon inversion formula can also be written as

$$f(x) = \frac{1}{8\pi^3} \int_{\|\beta\|=1} \lim_{\alpha' \rightarrow 0} \int_{-\infty}^{\infty} H_{\alpha'}(x \cdot \beta - t) F(t, \beta) dt d\beta . \quad (10)$$

We begin by finding a useful expression for the one-dimensional Fourier transform of $F(l, \beta)$ as a function of its first variable.

Lemma 2. (Ludwing [6], pp.52) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be such that $g'(t)$ is continuous and vanishes at infinity together with $g(t)$ faster than $\frac{1}{|t|}$; then

$$[\hat{g}(\omega) \operatorname{sgn} \omega]^{\vee}(t) = i(\mathcal{H}g)(t), \quad -\infty < t < \infty ,$$

where $(\hat{f})^{\vee}(t) \equiv f(t)$ and $i = \sqrt{-1}$.

For completeness, Ludwing's proof is reproduced in the Appendix, following Section 3.

The main result of this section is now stated in the following

Theorem 1. If $\frac{\partial^4 \tilde{f}}{\partial l^4}(l, \beta)$ is continuous, $\frac{\partial^{(i)}}{\partial l^{(i)}} \tilde{f}(l, \beta)$, $i = 0, 1, 2, 3, 4$ and $\frac{\partial^{(j)} F}{\partial l^{(j)}}(l, \beta)$, $j = 0, 1, 2$ vanish at infinity faster than $\frac{1}{|l|}$, then the density function $f(x)$ can be written as

$$f(x) = \frac{1}{8\pi^3} \int_{\|\beta\|=1} \lim_{\alpha' \rightarrow 0} \int_{-\infty}^{\infty} H_{\alpha'}(x \cdot \beta - t) F(t, \beta) dt d\beta .$$

Proof. By Lemma 2, with $g(l) = \frac{\partial F}{\partial l}(l, \beta)$ and $l = x \cdot \beta$,

$$\begin{aligned} \left(\mathcal{H} \frac{\partial F}{\partial l} \right) (l, \beta) &= \left[\frac{\widehat{\partial F}}{\partial l}(\omega, \beta) (-i \operatorname{sgn} \omega) \right]^{\vee} (l, \beta) \\ &= \left[i\omega \hat{F}(\omega, \beta) (-i \operatorname{sgn} \omega) \right]^{\vee} (l, \beta) , \end{aligned}$$

and

$$\left(\mathcal{H} \frac{\partial F}{\partial l} \right) (l, \beta) = \left[\omega \hat{F}(\omega, \beta) \operatorname{sgn} \omega \right]^{\vee} (l, \beta) . \quad (11)$$

Notice that this formula requires the continuity of $\frac{\partial^2 F}{\partial l^2}$ which in turn requires the continuity of $\frac{\partial^4 \tilde{f}}{\partial l^4}$ according to the definition of Hilbert transform, given by equation (2). Since $F(l, \beta) = \left(\mathcal{H} \frac{\partial \tilde{f}}{\partial l} \right) (l, \beta)$, using Lemma 2 again,

$$F(l, \beta) = \left[\frac{\partial \tilde{f}}{\partial l} (\omega, \beta) (-i \operatorname{sgn} \omega) \right]^{\vee} (l, \beta).$$

Applying Fourier transform,

$$\hat{F}(\omega, \beta) = \left[\frac{\partial \tilde{f}}{\partial l} (\omega, \beta) (-i \operatorname{sgn} \omega) \right]$$

and replacing this in equation (11), we get

$$\begin{aligned} \left(\mathcal{H} \frac{\partial F}{\partial l} \right) (l, \beta) &= \left[-i \omega \frac{\partial \tilde{f}}{\partial l} (\omega, \beta) \right]^{\vee} (l, \beta) \\ &= \left[-\frac{\partial^2 \tilde{f}}{\partial l^2} (\omega, \beta) \right]^{\vee} (l, \beta) \end{aligned}$$

and

$$\left(\mathcal{H} \frac{\partial F}{\partial l} \right) (l, \beta) = -\frac{\partial^2 \tilde{f}}{\partial l^2} (l, \beta). \tag{12}$$

On the other hand, recalling that $l = x \cdot \beta$,

$$\left(\mathcal{H} \frac{\partial F}{\partial l} \right) (l, \beta) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{1}{x \cdot \beta - t} \frac{\partial F}{\partial t} (t, \beta) dt.$$

By Lemma 1, with $g(t) = F(t, \beta)$, we get

$$\left(\mathcal{H} \frac{\partial F}{\partial l} \right) (l, \beta) = \frac{1}{\pi} \lim_{\alpha' \rightarrow 0} \int_{-\infty}^{\infty} H_{\alpha'}(x \cdot \beta - t) F(t, \beta) dt. \tag{13}$$

and from equations (12) and (13) we obtain

$$-\frac{\partial^2 \tilde{f}}{\partial l^2} (l, \beta) = \frac{1}{\pi} \lim_{\alpha' \rightarrow 0} \int_{-\infty}^{\infty} H_{\alpha'}(x \cdot \beta - t) F(t, \beta) dt. \tag{14}$$

Finally, replacing equation (14) in equation (1), we obtain the new inversion formula.

3. Stability Analysis

The purpose of this section is to define and analyze a stable method for the three-dimensional reconstruction procedure based on the inversion formulas of Section 2, given by

$$f(x) = \frac{1}{8\pi^3} \int_{\|\beta\|=1} \lim_{\alpha' \rightarrow 0} \int_{-\infty}^{\infty} H_{\alpha'}(x \cdot \beta - l) F(l, \beta) dl d\beta ,$$

and

$$F(l, \beta) = \frac{1}{\pi} \lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} H_{\alpha}(l - t) \tilde{f}(t, \beta) dt ,$$

when the data is not known exactly. Our method consists of eliminating the limit procedure -regularization by truncation in the formulas above, i.e., we define

$$f_{\alpha', \alpha}(x) = \frac{1}{8\pi^3} \int_{\|\beta\|=1} \int_{-\infty}^{\infty} H_{\alpha'}(x \cdot \beta - l) F_{\alpha}(l, \beta) dl d\beta , \quad (15)$$

and

$$F_{\alpha}(l, \beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} H_{\alpha}(l - t) \tilde{f}(t, \beta) dt . \quad (16)$$

The following two Lemmas are fundamental for our analysis.

Lemma 3. (Consistency). Let $g : \mathbb{R} \rightarrow \mathbb{R}$ with $g''(l)$ continuous and such that $g^{(i)}$, $i = 0, 1, 2$, vanish at infinity faster than $\frac{1}{|t|}$. If $G(l) = (\mathcal{I}g')(l)$ and

$$G_{\delta}(l) = \frac{1}{\pi} \int_{-\infty}^{\infty} g(t) H_{\delta}(l - t) dt, \quad 0 < \delta < 1 , \quad (17)$$

then

$$\|G_{\delta} - G\|_{\infty} \leq \frac{\delta M}{\pi} \left[4|\log \delta| + \frac{10}{3} \right] , \quad (18)$$

where $\|g'\|_{\infty} \leq M$.

Proof. By the definition of the kernel H_δ in equation (3), we get

$$G_\delta(l) = \frac{1}{\pi} \left[\frac{1}{\delta^2} \int_{l-\delta}^{l+\delta} g(t)dt - \int_{-\infty}^{l-\delta} \frac{1}{(l-t)^2} g(t)dt - \int_{l+\delta}^{\infty} \frac{1}{(l-t)^2} g(t)dt \right].$$

An integration by parts yields

$$G_\delta(l) = \frac{1}{\pi} \left[\frac{1}{\delta^2} \int_{l-\delta}^{l+\delta} g(t)dt - \left(\frac{g(l-\delta) + g(l+\delta)}{\delta} \right) + \int_{-\infty}^{l-\delta} \frac{1}{l-\delta} g'(t)dt + \int_{l+\delta}^{\infty} \frac{1}{l-t} g'(t)dt \right],$$

and using equation (7) with $\alpha = \delta$, we have

$$G_\delta(l) = \frac{1}{\pi} \left[0(\delta) + \int_{-\infty}^{l-\delta} \frac{1}{l-t} g'(t)dt + \int_{l+\delta}^{\infty} \frac{1}{l-t} g'(t)dt \right].$$

After a new integration by parts, this expression becomes

$$G_\delta(l) = \frac{1}{\pi} \left[0(\delta) + (g'(l+\delta) - g'(l-\delta)) \log \delta + \int_{-\infty}^{l-\delta} \log |l-t| g''(t)dt + \int_{l+\delta}^{\infty} \log |l-t| g''(t)dt \right]. \tag{19}$$

Subtracting equation (5) from equation (19), we obtain

$$G_\delta(l) - G(l) = \frac{1}{\pi} \left[0(\delta) + (g'(l+\delta) - g'(l-\delta)) \log \delta - \int_{l-\delta}^{l+\delta} \log |l-t| g''(t)dt \right].$$

Thus,

$$|G_\delta(l) - G(l)| \leq \frac{1}{\pi} \left[0(\delta) + |g'(l+\delta) - g'(l-\delta)| |\log \delta| + \int_{l-\delta}^{l+\delta} |\log |l-t|| |g''(t)|dt \right].$$

Now, since $0 < \delta < 1$, it follows that $|l-t| < 1$ and $\log |l-t| < 0$ for $l-\delta < t < l+\delta$. Hence, $\int_{l-\delta}^{l+\delta} |\log |l-t|| dt = -\int_{l-\delta}^{l+\delta} \log |l-t| dt = 2\delta(1 - \log \delta)$.

This result, plus the mean value theorem and the estimate in equation (7), allow us to write

$$\begin{aligned} |G_\delta(l) - G(l)| &\leq \frac{1}{\pi} \left[\frac{4}{3} M\delta + 2M\delta |\log \delta| + 2M\delta(1 - \log \delta) \right] \\ &= \frac{\delta M}{\pi} \left[\frac{10}{3} + 4|\log \delta| \right]. \end{aligned}$$

In what follows, ε denotes an upper bound for the amount of noise in the data, g^ε indicates the noisy data function and G_δ^ε represents the corresponding associate solution function obtained using formula (17).

Lemma 4. (Stability). Let $g, g^\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions that vanish at infinity faster than $\frac{1}{|t|}$. If $\|g - g^\varepsilon\|_\infty \leq \varepsilon$ then

$$\|G_\delta - G_\delta^\varepsilon\|_\infty \leq \frac{4\varepsilon}{\pi\delta}.$$

Proof. From equation (17) we readily have

$$\begin{aligned} |G_\delta(l) - G_\delta^\varepsilon(l)| &\leq \frac{1}{\pi} \left| \int_{-\infty}^{\infty} (g(t) - g^\varepsilon(t)) H_\delta(l-t) dt \right| \\ &\leq \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} |H_\delta(l-t)| dt \\ &= \frac{\varepsilon}{\pi} \frac{4}{\delta}. \end{aligned}$$

The following proposition will show that in the absence of noise in the data, the approximate solution $f_{\alpha',\alpha}$, obtained with the new method, is uniformly close to the exact density function f .

Theorem 2. (Consistency of the new method.) Under the hypothesis of Theorem 1,

$$\begin{aligned} \|f_{\alpha',\alpha} - f\|_\infty &\leq \frac{1}{2\pi^2} \left[M_2\alpha' \left(4|\log \alpha'| + \frac{10}{3} \right) \right. \\ &\quad \left. + \frac{4M_1\alpha}{\pi\alpha'} \left(4|\log \alpha| + \frac{10}{3} \right) \right], \end{aligned} \tag{20}$$

where $\left\| \frac{\partial^2 f}{\partial l^2} \right\|_\infty \leq M_1$ and $\left\| \frac{\partial^2 F}{\partial l^2} \right\|_\infty \leq M_2$, for all $(l, \beta) \in \mathbb{R} \times S^2$, $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$.

Proof. By Lemma 3, given β , identifying δ with α and $g(t)$ with $\tilde{f}(t, \beta)$, we get $\|F_\alpha(\cdot, \beta) - F(\cdot, \beta)\|_\infty \leq \frac{\alpha}{\pi} M(\beta) \left(4|\log \alpha| + \frac{10}{3}\right)$, where $\left\|\frac{\partial^2 \tilde{f}}{\partial t^2}(\cdot, \beta)\right\|_\infty \leq M(\beta)$. Since $\frac{\partial^2 \tilde{f}}{\partial t^2}(l, \beta)$ is uniformly bounded in $\mathbb{R} \times S^2$, it follows that

$$\|F_\alpha - F\|_\infty \leq \frac{\alpha}{\pi} M_1 \left(4|\log \alpha| + \frac{10}{3}\right),$$

and we can write for all $\beta \in S^2$,

$$F_\alpha(l, \beta) = F(l, \beta) + \sigma_\alpha(l), \tag{21}$$

where $\sigma_\alpha(l) = \sigma(l) \frac{\alpha}{\pi} M_1 \left(4|\log \alpha| + \frac{10}{3}\right)$, $-1 \leq \sigma(l) \leq 1$, $-\infty < l < \infty$.

Next, using relation (21), we rewrite equation (10) and (15) as follows:

$$f(x) = \frac{1}{8\pi^2} \int_{\|\beta\|=1} \left[\frac{1}{\pi} \lim_{\alpha' \rightarrow 0} \int_{-\infty}^{\infty} H_{\alpha'}(x \cdot \beta - l, \beta) F(l, \beta) dl \right] d\beta, \tag{22}$$

and

$$f_{\alpha', \alpha}(x) = \frac{1}{8\pi^2} \int_{\|\beta\|=1} \left[\frac{1}{\pi} \int_{-\infty}^{\infty} H_{\alpha'}(x \cdot \beta - l, \beta) F(l, \beta) dl + \frac{1}{\pi} \int_{-\infty}^{\infty} H_{\alpha'}(x \cdot \beta - l, \beta) \sigma_\alpha(l) dl \right] d\beta. \tag{23}$$

We estimate the difference between the inner integral in equation (22) and the first inner integral in equation (23) by applying Lemma 3 again, with $\delta = \alpha'$ and $g(l) = F(l, \beta)$. We get

$$\|G_{\alpha'} - G\|_\infty \leq \alpha' \frac{M_2}{\pi} \left[4|\log \alpha'| + \frac{10}{3}\right], \tag{24}$$

using the fact that $\left\|\frac{\partial^2 F}{\partial l^2}(l, \beta)\right\|_\infty \leq M_2$, for all $(l, \beta) \in \mathbb{R} \times S^2$. Here,

$$G_{\alpha'}(x \cdot \beta, \beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} H_{\alpha'}(x \cdot \beta - l, \beta) F(l, \beta) dl,$$

and

$$G(x \cdot \beta, \beta) = \lim_{\alpha' \rightarrow 0} G_{\alpha'}(x \cdot \beta, \beta).$$

Subtracting equation (23) from equation (22), we get the estimate

$$|f(x) - f_{\alpha',\alpha}(x)| \leq \frac{1}{8\pi^2} \int_{\|\beta\|=1} \|G_{\alpha'} - G\|_{\infty} d\beta \\ + \frac{1}{8\pi^2} \int_{\|\beta\|=1} \frac{1}{\pi} \int_{-\infty}^{\infty} |H_{\alpha'}(x \cdot \beta - l, \beta)| |\sigma_{\alpha}(l)| dl d\beta .$$

Using the estimate (24), the definition of $\sigma_{\alpha}(l)$ and evaluating

$$\int_{-\infty}^{\infty} |H_{\alpha'}(x \cdot \beta - l, \beta)| dl ,$$

we have

$$|f(x) - f_{\alpha',\alpha}(x)| \leq \frac{1}{8\pi^2} \left[\frac{M_2\alpha'}{\pi} \left(4|\log \alpha'| + \frac{10}{3} \right) \right. \\ \left. + \frac{M_1\alpha}{\pi^2} \cdot \left(4|\log \alpha| + \frac{10}{3} \right) \frac{4}{\alpha'} \right] \int_{\|\beta\|=1} d\beta ,$$

and

$$\|f - f_{\alpha',\alpha}\|_{\infty} \leq \frac{1}{2\pi^2} \left[M_2\alpha' \left(4|\log \alpha'| + \frac{10}{3} \right) + \frac{4M_1\alpha}{\pi\alpha'} \left(4|\log \alpha| + \frac{10}{3} \right) \right] .$$

The consistency estimate (20) shows that the first term is independent of α and tends to zero when α' decreases, while the second term increases without bound for fixed α . The choice $\alpha' = 0(\sqrt{\alpha})$ gives the uniform rate of convergence $\|f - f_{\alpha',\alpha}\|_{\infty} = 0(\sqrt{\alpha} |\log \alpha|)$ as $\alpha \rightarrow 0$.

The following proposition will show that attempting to reconstruct the approximate Radon inverse function $f_{\alpha',\alpha}$ is a stable problem with respect to perturbations in the data function \tilde{f} , in the maximum norm and for fixed α' and α . As in Lemma 4, ε denotes an upper bound for the amount of noise in the data, \tilde{f}^{ε} indicates the noisy data function and $f_{\alpha',\alpha}^{\varepsilon}$ represents the corresponding associated density function obtained using formulas (15) and (16). Notice that we only need continuity of the perturbed data function for the application of the new method.

Theorem 3. (Stability of the new method). If $\|\tilde{f} - \tilde{f}^{\varepsilon}\|_{\infty} \leq \varepsilon$, then

$$\|f_{\alpha',\alpha} - f_{\alpha',\alpha}^{\varepsilon}\| \leq \frac{8\varepsilon}{\pi^4\alpha\alpha'} .$$

Proof. We apply Lemma 4 with $g(t) = \tilde{f}(t, \beta)$, $\delta = \alpha$ and β fixed. With $\|\tilde{f} - \tilde{f}^{\varepsilon}\|_{\infty} \leq \varepsilon$ we obtain

$$\|F_{\alpha} - F_{\alpha}^{\varepsilon}\|_{\infty} \leq \frac{4\varepsilon}{\pi\alpha} .$$

Applying the same Lemma again, with $g(t) = F_\alpha(t, \beta)$, the perturbation $4\varepsilon/\pi\delta$ and $\delta = \alpha'$, we get, after integrating over the unit sphere,

$$\|f_{\alpha', \alpha} - f_{\alpha', \alpha}^\varepsilon\|_\infty \leq \frac{8\varepsilon}{\pi^4 \alpha', \alpha}. \quad (25)$$

We observe that we have restored stability with respect to the data. In fact, for fixed α' and α as $\varepsilon \rightarrow 0$, $f_{\alpha', \alpha}^\varepsilon \rightarrow f_{\alpha', \alpha}$ uniformly.

By using the triangular inequality, we have the following error estimate.

Theorem 4. (Convergence of the new method). Under the conditions of Theorems 2 and 3.

$$\begin{aligned} \|f_{\alpha', \alpha}\|_\infty &\leq \frac{1}{2\pi^2} \left[M_2 \alpha' \left(4|\log \alpha'| + \frac{10}{3} \right) \right. \\ &\quad \left. + \frac{4M_1 \alpha}{\pi \alpha'} \left(4|\log \alpha| + \frac{10}{3} \right) \right] + \frac{8\varepsilon}{\pi^4 \alpha', \alpha}. \end{aligned}$$

Remarks.

1. With the choice $\alpha = 0(\sqrt{\varepsilon})$, we obtain uniform convergence with rate $0(\varepsilon^{1/4} |\log \varepsilon|)$, provided that $\alpha' = 0(\sqrt{\alpha})$ as in Theorem 2.
2. The three dimensional image reconstruction from cone-beam projections leads naturally, in the case of radial symmetry, to the study of Abel's type of integral equations. For a complete analysis and the numerical implementation of the ideas discussed in this paper, applied to the one-dimensional case, the reader is referred to Murio et al [9].

Appendix.

Lemma 2, Section 2. (Ludwig [6], pp. 52).

Let $g(t)$, $g : \mathbb{R} \rightarrow \mathbb{R}$ be such that $g'(t)$ is continuous and vanishes at infinity together with $g(t)$ faster than $\frac{1}{|t|}$; then

$$[\hat{g}(\omega) \operatorname{sgn} \omega]^\vee(t) = i(\mathcal{H}g)(t),$$

$-\infty < t < \infty$, where $(\hat{f})^\vee(t) \equiv f(t)$ and $i = \sqrt{-1}$.

Proof. Define

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s) \operatorname{sgn} \omega e^{i\omega(t-s)} ds d\omega. \quad (26)$$

Thus,

$$\begin{aligned} h(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \operatorname{sgn} \omega \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s) e^{-i\omega s} ds \right) e^{i\omega t} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \operatorname{sgn} \omega \hat{g}(\omega) e^{i\omega t} d\omega, \end{aligned}$$

and

$$h(t) = [\hat{g}(\omega) \operatorname{sgn} \omega]^{\vee}(t). \quad (27)$$

Equation (27) states that $h(t)$ is the one-dimensional inverse Fourier transform of the function $\hat{g}(\omega) \operatorname{sgn} \omega$.

On the other hand, from the definition (26), we have

$$h(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s) \operatorname{sgn} \omega \sin(\omega(t-s)) ds d\omega.$$

After integrating by parts, recalling that $\int_{-\infty}^{\infty} g'(s) ds = 0$ we can write

$$\begin{aligned} h(t) &= \frac{i}{\pi} \int_0^1 \int_{-\infty}^{\infty} -g'(s) \frac{\cos \omega(t-s) - 1}{\omega} ds d\omega \\ &\quad + \frac{i}{\pi} \int_1^{\infty} \int_{-\infty}^{\infty} -g'(s) \frac{\cos \omega(t-s)}{\omega} ds d\omega. \end{aligned}$$

Interchanging the order of integration, we have

$$h(t) = \frac{i}{\pi} \int_{-\infty}^{\infty} -g'(s) \int_0^{\infty} \frac{\cos \omega(t-s) - H(1-\omega)}{\omega} d\omega ds,$$

where

$$H(1-\xi) = \begin{cases} 0, & \xi > 1, \\ 1, & \xi < 1. \end{cases}$$

If $t-s \neq 0$, we set $\Omega = \omega|t-s|$. Thus,

$$\begin{aligned} h(t) &= \frac{i}{\pi} \int_{-\infty}^{\infty} -g'(s) \left[\frac{\cos \Omega - H(1-\Omega)}{\Omega} d\Omega \right. \\ &\quad \left. + \int_0^1 \frac{H(1-\Omega) - H\left(1 - \frac{\Omega}{|t-s|}\right)}{\Omega} d\Omega \right] ds \\ &= \frac{i}{\pi} \int_{-\infty}^{\infty} -g'(s) [c - \log |t-s|] ds. \end{aligned}$$

with c constant.

Hence,

$$\begin{aligned} h(t) &= \frac{i}{\pi} \int_{-\infty}^{\infty} g'(s) \log |t-s| ds \\ &= \frac{i}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|t-s| \geq \epsilon} \frac{g(s)}{t-s} ds \end{aligned}$$

and

$$h(t) \equiv 1(\mathcal{H})(t). \quad (28)$$

Finally, from equations (27) and (28) we have $[\hat{g}(\omega) \operatorname{sgn} \omega]^{\vee}(t) = i(\mathcal{H})(t)$.

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