A GENERALIZED TIKHONOV REGULARIZATION USING TWO PARAMETERS APPLIED TO LINEAR INVERSE ILL-POSED PROBLEMS

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Abstract. In this paper we consider a generalization of the Tikhonov Regularization for the linear ill-posed operator equations $Ax = y$ using the functional

$$M_a^e[A,x,y^d,R] = \|Ax - y^d\|^2 + a\|Rx\|^2 + e\|x\|^2,$$

where $\varepsilon > 0, \alpha > 0$ and can be a lineal or nonlinear operator.

We develop existence, stability, convergence results and error estimates of the approximated solutions to the problem in a Hilbert space setting. We applied the method to some particular Fredholm integral equations of first kind.

1 Introduction

In this paper we consider the generalization Tikhonov regularization for the operator equation

$$Ax = y \tag{1}$$

where $A : X \to X$ is a compact operator between Hilbert spaces

Many inverse problems can be modeled by a first Fredholm integral equation

$$(Kx)(s) = \int_\Omega k(s,t)x(t)dt = g(s)$$

where the kernel function k is a continuous and smooth function over a compact domain Ω. If K is nor degenerate then it is well known that the solution of (1) does not depend continuously of g in and we deal with an ill-posed problem. Therefore regularization techniques are needed in order to stabilize the approximated solutions. The regularization Tikhonov technique is based in the used of two regularization parameters

By using an appropriate operator R, which depends on prior knowledge of the exact solution x of (1).

2 Linear ILL-Posed Problems

The problem (1) is an ill posed in the sense that the solutions do not de-

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pend continuously on the data. The generalized solution of the problem (1) is $\hat{x}$ where ever

$$y \in D(A^{\dagger}) = R(A) \oplus R(A)^{\perp}$$

the domain of the Moore-Penrose inverse $A^{\dagger}$ of A. It is well known that if $R(A)$ is not closed, the operator $A^{\dagger}$ is not bounded and then the problem (1) is ill-posed. In general, the data y may not be available exactly, instead of we han an approximation y^{ε} such that

$$\left\| y - y^{\varepsilon} \right\| \leq \varepsilon.$$

In order to find an approximate solution to the generalized solution, we consider the minimization problem

$$\min_{x \in D(R)} M^{\varepsilon}_{\alpha,\beta}\left[x, x^*, A, R, y^{\varepsilon}\right] \quad (P_{\alpha,\beta}) \quad (2)$$

where

$$M_{\alpha,\beta}\left[x, y^{\varepsilon}\right] = M^{\varepsilon}_{\alpha,\beta}\left[x, x^*, A, R, y^{\varepsilon}\right]$$
$$= \left\| A(x) - y^{\varepsilon} \right\|_Z^2 + \alpha \left\| x - x^* \right\|_X^2 + \beta \left\| R(x) \right\|_Y^2,$$

and R is a regularized operator $R : D(R) \subset X \to Y$ be a densely defined with closed range. $\alpha, \beta \geq 0$, $x \in D(R)$

The Tikhonov generalized solution that depends of two parameter is the minimize of the problem (2) given by

$$x^{\varepsilon}_{\alpha,\beta} = \arg\min_{x \in D(R)} \left\{ \begin{array}{l} \left\| A(x) - y^{\varepsilon} \right\|_Z^2 + \\ \alpha \left\| x - x^* \right\|_X^2 + \beta \left\| R(x) \right\|_Y^2 \end{array} \right\}$$
$$= (A^*A + \alpha I + \beta R^*R)^{-1} A^* y^{\varepsilon}$$

The paper is organized as follows. In section 3 we present our main results on stability and convergence rates. In section 4 we apply our results to several examples related with Fredholm equations of first kind.

3 Convergence Analisis

Theorem 1: (Convergence-Stability).

(i) Assume that the problem $A(x) = y$ has a solution and suppose

$$\|y - y^{\varepsilon_n}\|$$

$$\leq \varepsilon_n \to 0,\ \ \alpha_n \to 0,\ \ \beta_n \to 0,\ \ \frac{\varepsilon_n^2}{\alpha_n} \to 0$$

and $\dfrac{\varepsilon_n}{\alpha_n} \to 0$

Then the sequence $\{x^{\varepsilon_n}_{\alpha_n,\beta_n}\}$ has a subsequence $\{x_{nk}\}$ converging strongly to a x^*-m.n.s., , w such that $A(w) = y$ and $A(x_{nk}) \to y$ and $R(x_{nk}) \to R(w)$.

(ii) Assume that the problem $A(x) = y$ has a solution and suppose

$$\|y - y^{\varepsilon_n}\| \leq \varepsilon_n \to 0,$$

$$\alpha_n \to 0,\ \ \beta_n \to 0,\ \ \frac{\varepsilon_n^2}{\alpha_n} \to 0 \text{ and } \frac{\alpha_n}{\beta_n} \to 0$$

Then the sequence $\{x^{\varepsilon_n}_{\alpha_n}\}$ has a weakly converging subsequence $\{x_{nk}\}$ and every weak limit x of $\{x_{nk}\}$ satisfies, $A(w) = y$

$$R(x_{nk}) \to R(w), \text{and. } A(x_{nk}) \to y$$

(iii) Assume that the problem has a solution and suppose

$$\|y - y^{\varepsilon_n}\| \leq \varepsilon_n \to 0,$$

$$\alpha_n - \beta_n \to 0,\ \ \frac{\varepsilon_n^2}{\alpha_n} \to 0$$

Then the sequence $\{x^{\varepsilon_n}_{\alpha_n}\}$ has a subsequence $\{x_{nk}\}$ converging strongly to a w such that , $A(w) = y, R(x_{nk}) \to R(w)$, and $A(x_{nk}) \to y$. Moreover $(w, \psi(w)$ is a $(x^*, R(x^*))$-m.n.s.

Proof: Part 1. Let $x_{\alpha_n}^{\varepsilon_n}$ the solution of the problem $(P_{\alpha,\beta})$, then

$$\left\|A(x_{\alpha_n,\beta_n}^{\varepsilon_n})-y^{\varepsilon}\right\|_Z^2+\alpha\left\|x_{\alpha_n,\beta_n}^{\varepsilon_n}-x^*\right\|_X^2$$
$$+\beta\left\|R(x_{\alpha_n,\beta_n}^{\varepsilon_n})\right\|_Y^2$$
$$\leq\left\|A(x)-y^{\varepsilon}\right\|_Z^2+\alpha\left\|x-x^*\right\|_X^2+\beta\left\|R(x)\right\|_Y^2$$

for all $x\in D(R)$.

Let $x_o\in D(R)$ such that $A(x_o)=y$. *Then*

$$\left\|A(x_{\alpha_n,\beta_n}^{\varepsilon_n})-y^{\varepsilon_n}\right\|_Z^2+\alpha_n\left\|x_{\alpha_n,\beta_n}^{\varepsilon_n}-x^*\right\|_X^2$$
$$+\beta_n\left\|R(x_{\alpha_n,\beta_n}^{\varepsilon_n})\right\|_Y^2$$
$$\leq\left\|A(x_o)-y^{\varepsilon}\right\|_Z^2+\alpha_n\left\|x_o-x^*\right\|_X^2$$
$$+\beta_n\left\|R(x_o)\right\|_Y^2 \qquad (3)$$
$$\leq\varepsilon^2+\alpha_n\left\|x_o-x^*\right\|_X^2+\beta_n\left\|R(x_o)\right\|_Y^2$$

Since the sequence $\left\{\left(x_{\alpha_n,\beta_n}^{\varepsilon_n},R(x_{\alpha_n,\beta_n}^{\varepsilon_n}),A(x_{\alpha_n,\beta_n}^{\varepsilon_n})\right)\right\}$ is bounded in $X\times X\times Y$, there is a subsequence $\left\{x_{\alpha_{nk},\beta_{nk}}^{\varepsilon_{nk}}\right\}$ of $\left\{x_{\alpha_n,\beta_n}^{\varepsilon_n}\right\}$ such that

$$x_{\alpha_{nk},\beta_{nk}}^{\varepsilon_{nk}}\xrightarrow{w}w$$
$$R(x_{\alpha_{nk},\beta_{nk}}^{\varepsilon_{nk}})\xrightarrow{w}R(w)$$
$$A(x_{\alpha_{nk},\beta_{nk}}^{\varepsilon_{nk}})\xrightarrow{w}A(w)$$

According to (3) we have that $\left\|A(x_{\alpha_n,\beta_n}^{\varepsilon_n})-y^{\varepsilon}\right\|\to 0$ and therefore

$A(x^{\varepsilon_n}_{\alpha_n,\beta_n}) \to y$ as $n \to \infty$. Since A is weakly closed we have $y = F(w)$. Also for (3) we have that

$$\left\| x^{\varepsilon_n}_{\alpha_n,\beta_n} - x^* \right\|^2_X \le \frac{\varepsilon_n^{\ 2}}{\alpha_n} + \left\| x_o - x^* \right\|^2_X + \frac{\beta_n}{\alpha_n} \left\| R(x_o) \right\|$$

Then

$$\overline{lim} \left\| x^{\varepsilon_n}_{\alpha_n,\beta_n} - x^* \right\|^2_X \le \left\| x_o - x^* \right\|^2_X .$$

Therefore w is a $x^*\_m.n.s.$ replacing x_o for w in the last inequality we have that

$$\overline{lim} \left\| x^{\varepsilon_n}_{\alpha_n,\beta_n} - x^* \right\|^2_X \le \left\| w - x^* \right\|^2_X$$

This implies that $x^{\varepsilon_{nk}}_{\alpha_{nk},\beta_{nk}} \longrightarrow w$ and clearly $A(x^{\varepsilon_{nk}}_{\alpha_{nk},\beta_{nk}}) \longrightarrow A(w)$ and $R(x^{\varepsilon_{nk}}_{\alpha_{nk},\beta_{nk}}) \longrightarrow R(w)$.

The proof of Part 2 Using part of the part 1 we have that

$$\begin{aligned}
\left\| R(w) \right\| &\le \underline{lim} \left\| R(x^{\varepsilon_{nk}}_{\alpha_{nk},\beta_{nk}}) \right\| \\
&\le \overline{lim} \left\| R(x^{\varepsilon_{nk}}_{\alpha_{nk},\beta_{nk}}) \right\| \\
&\le \overline{lim} \left(\frac{\varepsilon_n^{\ 2}}{\beta_n} + \frac{\varepsilon_n}{\beta_n} \left\| x_o - x^* \right\|^2_X + \left\| R(w) \right\| \right) \\
&\le \left\| R(w) \right\|
\end{aligned}$$

This implies that $R(x^{\varepsilon_n}_{\alpha_n,\beta_n}) \to R(w)$ and $A(x^{\varepsilon_n}_{\alpha_n,\beta_n}) \to y$.

The part 3 is proved in the similar way as part 2.

Theorem 4: (Rate of Convergence). *Let's suppose* $\|y - y^{\varepsilon}\| \le \varepsilon$ *and* $A(x_o) = y$ *and there is* μ *such that* $A^{*}\mu = x^{*} - x_0 + R^{*}v$, where $v = \psi^{*} - R(x_0)$. *Taking* $\alpha = \beta = \varepsilon$ *we have that*

$$\|x_\alpha - x_o\| = O(\alpha)$$

Proof: Taking $x = x_o$ and $\|y - y^{\varepsilon}\| \le \varepsilon$,

$$\begin{aligned}
&\|A(x_\alpha) - y^{\varepsilon}\|_Z^2 + \alpha\|x_{\alpha_n} - x^{*}\|_X^2 \\
&+\beta\|R(x_{\alpha_n}) - \psi^{*}\|_Y^2 \\
&\le \|A(x_o) - y^{\varepsilon}\|_Z^2 + \alpha\|x_o - x^{*}\|_X^2 \\
&+\beta\|R(x_o) - \psi^{*}\|_Y^2 \\
&\le \varepsilon^2 + \alpha\|x_o - x^{*}\|_X^2 + \beta\|R(x_o) - \psi^{*}\|_Y^2
\end{aligned}$$

Since

$$\begin{aligned}
\|x_\alpha - x^{*}\|_X^2 = \|x_\alpha - x_0\|_X^2 + \|x_o - x^{*}\|_X^2 \\
- 2(x_\alpha - x_0, x^{*} - x_0)
\end{aligned}$$

and

$$\begin{aligned}
&\|R(x_\alpha) - \psi^{*}\|_X^2 \\
&= \|R(x_\alpha) - R(x_o)\|_X^2 + \|R(x_o) - \psi^{*}\|_X^2 \\
&-2(R(x_\alpha) - R(x_0), \psi^{*} - R(x_0))
\end{aligned}$$

we have

$$\begin{aligned}
&\left\|A(x_\alpha)-y\right\|_Z^2+\alpha\left\|x_\alpha-x_o\right\|_X^2\\
&+\alpha\left\|R(x_{\alpha_n})-R(x_o)\right\|_Y^2\\
&\leq \varepsilon^2+2\alpha(x_\alpha-x_0,x^*-x_0)\\
&\ +2\alpha(R(x_\alpha)-R(x_0),\psi^*-R(x_0))\\
&=\varepsilon^2+2\alpha(x_\alpha-x_0,x^*-x_0)\\
&\ +2\alpha(R(x_\alpha-x_0),v)
\end{aligned}$$

$$\begin{aligned}
&=\varepsilon^2+2\alpha(x_\alpha-x_0,x^*-x_0)\\
&\ +2\alpha(x_\alpha-x_0,R^*v)\\
&=\varepsilon^2+2\alpha(x_\alpha-x_0,x^*-x_0+R^*v)\\
&=\varepsilon^2+2\alpha(x_\alpha-x_0,A^*\mu)\\
&=\varepsilon^2+2\alpha(A(x_\alpha-x_0),\mu)\\
&=\varepsilon^2+2\alpha(A(x_\alpha)-y,\mu)\\
&\leq \varepsilon^2+2\alpha\left\|A(x_\alpha)-y\right\|\left\|\mu\right\|.
\end{aligned}$$

Using the property of the real numbers, if $a^2\leq c^2+ab$ with $a,b,c>0$, then

$$a\leq b+c,$$

we have that

$$\left\|A(x_\alpha)-y\right\|_Z^2\leq \varepsilon^2+2\alpha\left\|A(x_\alpha)-y\right\|\left\|\mu\right\|.$$

Then

$$\left\|A(x_\alpha)-y\right\|_Z\leq \varepsilon+2\alpha\left\|\mu\right\|.$$

From the last estimate we have that

$$\alpha\left\|x_\alpha-x_o\right\|_Z^2\leq(\varepsilon+2\alpha\left\|\mu\right\|)^2$$

Then

$$\left\| x_\alpha - x_o \right\|_2 \le \frac{\varepsilon + 2\alpha \|\mu\|}{\sqrt{\alpha}} = \vartheta(\sqrt{\alpha})$$

4 Numerical Examples In this section we present some applications to some Fredholm equations of first kind using the regularization Tikhonov technique of two regularization parameters $\alpha, \beta > 0$. We will add 2 % random normal-distribution error to the exact data (to simulate the measurement errors), we find the approximated solution. We describe some examples and report the error in L_2 for a particular selection of values for the parameters α and β. We select, for each example, the best values for these parameters, the ones that result in the smallest solution error. In the figures, we have the exact solution, and we show the optimal α and β as well as the reconstructed images obtained using these optimal parameters.

Example 1. Considerer the Fredholm equations of first kind given by

$$(Ax)(s) = \int_0^1 k(s-t)x(t)dt = g(s) \quad 0 \le s \le 1,$$

where the kernel is the convolution type $k(s,t) = k(s-t)$, with

$$k(t) = \frac{1}{\sigma\sqrt{\pi}} exp(-(t-h/2)/\sigma^2)$$

and $\sigma = 0.05$ and $h = 0.0125$. We will apply the method of two parameters by discretization the linear inverse problem the equation reducing the problem to finite dimension by solving the linear problem. The solutions that depend of the parameters are given by

$$x^{\varepsilon}_{\alpha,\beta} = (A^*A + \alpha I + \beta B^*B)^{-1} A^*b.$$

The operator of Regularization is given by

$$R = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & & \\ & & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

See figure 1.

Example2. We consider in this case the kernel given by

$$k(s,t) = (\cos s + \cos t)/(\operatorname{sen} u / u)^2$$

with $u = \pi(\operatorname{sen} s + \operatorname{sen} t)$

where the exact solution is

$$x(t) = 2\exp(-6(-\pi/2 + t - 8)^2) + \exp(-2(-\pi/2 + t + 5)^2)$$

and the exact data is

$$g(s) = 2\exp(-6(s - 8)^2) + \exp(-2(s + 5)^2)$$

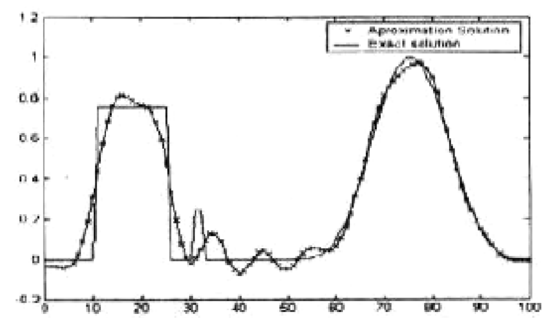

Figure 1.

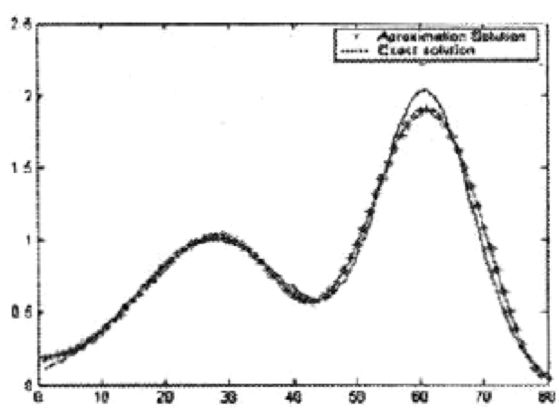

Figure 2. $\alpha = 0.00005$, $\beta = 0.001$ error=0.0465 rel_error=0.0185

5. References

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