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A simple test for asymptotic stability in some dynamical systems

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Abstract

In this paper we analyze asymptotic stability of the dynamical system $\dot{x} = f(x)$ defined by a $\mathcal{C}^1$ function $f : \mathcal{D} \to \mathbb{R}^n$ where $\mathcal{D} \in \mathbb{R}^n_+$ is an open set. We obtain a criterion of stability for the equilibrium solution $\bar{x} \in \mathcal{D}$ when the vector field f satisfies *a*) $\partial_i f_i(\bar{x}) < 0$ and *b*) $(\partial_i f_i(\bar{x}))^{-1}\partial_i f_j(\bar{x}) + (\partial_j f_j(\bar{x}))^{-1}\partial_j f_i(\bar{x}) > 2$ for $i, j = 1, \ldots, n$.

Keywords: ordinary differential equations, asymptotic stability, equilibrium solution.

1 Introduction

In 1892, A. M. Lyapunov developed his stability theory for nonlinear ordinary differential equations which characterizes the behavior of the dynamical systems trajectories in the sense that nearby solutions remain that way from now on (Hirsch and Smale, [9]). He established very useful stability criteria for dynamical systems of the form:

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a $\mathcal{C}^1$ map and $\mathcal{D} \subset \mathbb{R}^n$ is an open set.

The first Lyapunov method, also known as *Indirect Method of Lyapunov* (IML) allows to studying the stability of the equilibrium points for a dynamical system of the type (1) by analyzing the stability of the trivial solution for the linearized system:

$$\frac{dy(t)}{dt} = Df(\bar{x})y + G(y),$$

where $G(y) = O(\|y\|^2)$. Using the IML, it is possible prove that $y = 0$ is asymptotically stable if and only if $\Re(\lambda) < 0$ for any eigenvalue λ of the matrix $Df(\bar{x})$, and so unstable, if there exists an eigenvalue λ of the matrix $Df(\bar{x})$ with $\Re(\lambda) > 0$. We note that the IML does not allow us to obtain a conclusion if one of the eigenvalues λ of the matrix $Df(\bar{x})$ has real part zero, $\Re(\lambda) = 0$ (Khalil, [12]).

In this paper, we will consider the second Lyapunov method, also known as *Direct Method of Lyapunov* (DML), in which the stability of an equilibrium point $\bar{x}$ requires the flow associated with the dynamical system (1) being decreased on some scalar function V for which $\bar{x}$ is an isolated minimum. This function is known as the Lyapunov function.

For the Lyapunov function $V : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ where $\mathcal{D}$ containing the origin, and its orbital derivative $\dot{V} : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ defined by

$$\dot{V}(x) = DV(x)(f(x)),$$

the DML establishes:

1. If $V(x)$ is positive definite and $\dot{V}(x)$ is negative semi-definite, then the origin is stable.

2. If $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite, then the origin is asymptotically stable.

In general, for any equilibrium solution of (1) the DML states that:

Theorem 1. *Let $\bar{x} \in \mathcal{D}$ be an equilibrium of (1). Let $V : \mathcal{B} \to \mathbb{R}$ be a continuous function defined on a neigborhood $\mathcal{B} \subset \mathcal{D}$ of $\bar{x}$, differentiable on $\mathcal{B} - \bar{x}$, such that*

a) $V(\bar{x}) = 0$ *and* $V(x) > 0$ *if* $x \neq \bar{x}$*;*

b) $\dot{V}(x) \leq 0$ *in* $\mathcal{B} - \bar{x}$*,*

then $\bar{x}$ is stable. Furthemore, if

c) $\dot{V}(x) < 0$ *in* $\mathcal{B} - \bar{x}$*,*

then $\bar{x}$ is asymptotically stable.

In the twentieth century, the DML became in the principal tool to analyze global stability of dynamical systems applied to basic sciences and engineering. The main setback of this method is precisely to find a Lyapunov function, because there is not a systematic method for finding. The suggestion is to propose a function and check if this candidate satisfies the stability conditions (Perko, [20]).

While the intention of A. M. Lyapunov was to study movement stability (Taylor and Francis, [14]), the DML found a wide range of applications. For example, in problems related with automatic regulation and control of dynamical processes (Rouche et al., [21]; Vasilév, [25]; Yoshizawa, [26]; Artstein, [2]; Barbastin, [3]); in competition models (Goh, [7, 8]; Takeuchi, [23]); in SIR models (Mena-Lorca and Hethcote, [16]; Safi and Garba, [22]), in SIRS models (O'Regan et al., [19]); in models with two compartments (A. Yu, [1]), and in the proof of the Hopf bifurcation theorem (cited by O?Regan et al., [19]).

Recently, Lyapunov functions are being applied within the fractional calculus to analyze the stability of dynamical systems. In this field, the method is called

Fractional Lyapunov Direct Method (Yan Li et al., [13]; Momani and Hadid, [17]; Zhang et al., [27]; Tarasov, [24]). In 2011, it was used Lyapunov functions to analyze the dynamics of the Hopf bifurcation in a class of models that exhibit Zip bifurcation (Escobar and Gonzáles, [4], Giesl and Hafstein, [5, 6]). In 2012 the same authors designed an algorithm to explain the construction of these functions.

There are some systems where the Lyapunov function is defined in a natural way, like in the case of electrical or mechanical systems where *energy* is often a Lyapunov function. In mathematical biology, more precisely in population dynamic modeled through the mass action law, the functions of Goh type

$$V(x) = \sum_{i=1}^{n} a_i \left[x_i - \bar{x}_i - \bar{x}_i \ln\left(\frac{x_i}{\bar{x}_i}\right)\right], \tag{2}$$

where a_i for $i = 1, \dots, n$ are positive constants that satisfy the first item of Theorem 1 while the other items are reduced to find the constants a_i that will satisfy them.

B. S. Goh (Goh, [7]) used the function defined in (2) to prove global stability in mutualism models of the form

$$\dot{x}_i = x_i f_i(x_1, x_2, \dots, x_n) \quad i = 1, 2, \dots, n.$$

In this paper we establish global stability properties for the dynamical system (1) following the same ideas of S. B. Goh in (Goh, [7]). That is, we use the Lyapunov function (2) with specific values of the constants a_i to determine the stability conditions.

2 Calculus and linear algebra

Theorem 2. *Let E be an open subset of $\mathbb{R}^n$ containing x_0, if $f : E \subset \mathbb{R}^n \to \mathbb{R}$ such that $f \in \mathcal{C}^3(E)$, $f(x_0) = 0$ and Hessian matrix $Hf(x_0)$ is positive definite, then x_0 is a relative minimum of f. Similarly, if $Hf(x_0)$ is negative definite, then x_0 is a relative maximum of f.*

See [15] for proof of Theorem 2.

Theorem 3. *(Sylvester's Criterion). A real symmetric matrix is positive definite positive.*

See [10] for proof of Theorem 3.

3 Test of stability

In this section we establish a test for the asymptotic stability of the system (1) equilibrium when $\mathcal{D}$ is an open subset of

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, \dots, n\}.$$

The following proposition relates the equilibrium stability with the sign of certain determinants.

Proposition 1. *Let $\mathcal{D}$ be an open subset of $\mathbb{R}^n_+$ containing $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$. Suppose that the function $f : \mathcal{D} \to \mathbb{R}^n$ defined in (1) satisfies $f \in \mathcal{C}^1(\mathcal{D})$ and $f(\bar{x}) = 0$. Let $\Delta_j(\bar{x})$ be the determinants defined by*

$$\Delta_j(\bar{x}) = (-1)^j \left| \frac{a_j}{\bar{x}_j} \frac{\partial f_j(\bar{x})}{\partial x_i} + \frac{a_i}{\bar{x}_i} \frac{\partial f_i(\bar{x})}{\partial x_j} \right|_{i=1,\ldots,j}, \quad j = 1, \ldots, n \tag{3}$$

where a_j is a positive constant.

1. *If $\Delta_j(\bar{x})$ for $j = 1, \ldots, n$ are positive, then $\bar{x}$ is globally asymptotically stable.*
2. *If $\Delta_j(\bar{x})$ for $j = 1, \ldots, n$ has alternate signs starting with a negative value, then $\bar{x}$ is unstable.*

Proof. Let $a_1, \ldots, a_n$ be positive constants, for $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \mathcal{D}$, then the function defined in (2) satisfies the condition $V(\bar{x}) = 0$. On the other hand, the i-th term of (2) is:

$$\eta(x_i) = a_i \left[x_i - \bar{x}_i - \bar{x}_i \ln\left(\frac{x_i}{\bar{x}_i} \right) \right]. \tag{4}$$

Observe that

$$\eta'(x_i) = a_i \left(1 - \frac{\bar{x}_i}{x_i} \right),$$

Observe that which implies that $\eta'(x_i) > 0$ if and only if $\bar{x}_i < x_i$ and $\eta'(x_i) < 0$ if and only if $\bar{x}_i > x_i$. Thus $\bar{x}_i$ is a global minimum of η defined in (4). Since $\eta(\bar{x}_i) = 0$, then $\eta(x_i) > 0$ for all $x_i \neq \bar{x}_i$ therefore $V(x) > 0$ for all $x \neq \bar{x}$. From DML we conclude that if its orbital derivative is negative ($\dot{V}(x) < 0$) for all $x \in \mathcal{D} \backslash \{\bar{x}\}$, then $\bar{x}$ is asymptotically stable on $\mathcal{D}$, while $\bar{x}$ is unstable when $\dot{V}(x) > 0$ (*see* Theorem 1).

Observe that $\dot{V}(x) = -g(x)$ where

$$g(x) = \sum_{i=1}^{n} a_i \left(\frac{\bar{x}_i}{x_i} - 1 \right) f_i(x).$$

Since $g(\bar{x}) = 0$, then to prove the stability of $\bar{x}$ it is enough to verify that $\bar{x}$ is a minimum of g on $\mathcal{D}$, and any other equilibrium solution $y \in \mathcal{D}$ of (1) satisfies that $g(y) \geq g(\bar{x})$. The derivative of g is given by the gradient vector

$$\bigtriangledown g(x) = \left(\frac{\partial g(x)}{\partial x_1}, \ldots, \frac{\partial g(x)}{\partial x_n} \right),$$

where

$$\frac{\partial g(x)}{\partial x_k} = -a_k \frac{\bar{x}_k}{x_k^2} f_k(x) + \sum_{i=1}^{n} a_i \left(\frac{\bar{x}_i}{x_i} - 1 \right) \frac{\partial f_i(x)}{\partial x_k}, \tag{5}$$

for $k = 1, \ldots, n$. From (5) we have that $\partial g(\bar{x}) / \partial x_k = 0$ for $k = 1, \ldots, n$, which implies $\bigtriangledown g(\bar{x}) = 0$. Therefore $\bar{x}$ is a critical point of g.

The Hessian matrix of $g(x)$ is

$$Hg(x) = \left(\frac{\partial^2 g(x)}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,n}, \tag{6}$$

where

$$\begin{aligned}\frac{\partial^2 g(x)}{\partial x_j \partial x_k} &= -a_k \frac{\bar{x}_k}{x_k^2}\frac{\partial f_k(x)}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\sum_{i=1}^{n} a_i \left(\frac{\bar{x}_i}{x_i} - 1\right)\frac{\partial f_i(x)}{\partial x_k}\right] \\ &= -\left(a_k \frac{\bar{x}_k}{x_k^2}\frac{\partial f_k(x)}{\partial x_j} + a_j \frac{\bar{x}_j}{x_j^2}\frac{\partial f_j(x)}{\partial x_k}\right) + \sum_{i=1}^{n} a_i \left(\frac{\bar{x}_i}{x_i} - 1\right)\frac{\partial^2 f_i(x)}{\partial x_j \partial x_k},\end{aligned} \tag{7}$$

for $j, k = 1, \ldots, n$ and $j \neq k$. As a result

$$\frac{\partial^2 g(\bar{x})}{\partial x_j \partial x_k} = -\left(\frac{a_k}{\bar{x}_k}\frac{\partial f_k(\bar{x})}{\partial x_j} + \frac{a_j}{\bar{x}_j}\frac{\partial f_j(\bar{x})}{\partial x_k}\right). \tag{8}$$

Therefore, the determinant of Hessian matrix of $g(x)$ evaluated at $\bar{x}$ is:

$$\begin{aligned}|Hg(\bar{x})| &= \left|\frac{\partial^2 g(x)}{\partial x_i \partial x_j}\right|_{i,j=1,\ldots,n} \\ &= \left|-\left(\frac{a_j}{\bar{x}_j}\frac{\partial f_j(\bar{x})}{\partial x_i} + \frac{a_i}{\bar{x}_i}\frac{\partial f_i(\bar{x})}{\partial x_j}\right)\right|_{i,j=1,\ldots,n} \\ &= (-1)^j \left|\frac{a_j}{\bar{x}_j}\frac{\partial f_j(\bar{x})}{\partial x_i} + \frac{a_i}{\bar{x}_i}\frac{\partial f_i(\bar{x})}{\partial x_j}\right|_{i,j=1,\ldots,n} \\ &= \Delta_n(\bar{x}).\end{aligned}$$

Since $Hg(\bar{x})$ is a symmetric matrix, and assuming that all its principal minors are positive, then from Theorem 3 we have that $\bar{x}$ is a local minimum of g on $\mathcal{D}$. Now, suppose that $y \in \mathcal{D}$ is another equilibrium solution of (1) then $g(y) = g(\bar{x}) = 0$. Similarly it is verified that if its principal minors have alternating signs for $k = 1, \ldots, n$, starting with a negative value, then $\bar{x}$ is unstable, which completes the proof.

From the above proposition, the following corollary is derived:

Corollary 4. *If the Hessian matrix $Hg(x)$ defined in (6) evaluated at $\bar{x}$ is positive definite, then $\bar{x}$ is globally asymptotically stable on $\mathcal{D}$ and unstable when $Hg(\bar{x})$ is negative definite.*

The following theorem summarizes the main result of this work. The novelty of next test consists in replacing the expertise of the authors to find the constants a_i defined in (2) for conditions easy to verify.

Theorem 5 (**Stability Test**). *Let $\bar{x} \in \mathcal{D} \subset \mathbb{R}_+^n$ be an equilibrium solution of nonlinear system (1). If*

1. $\dfrac{\partial f_i(\bar{x})}{\partial x_i} < 0$ *for* $i = 1, 2, \ldots, n$. *and*

2. $l_{ij}(\bar{x}) > 2$ *for* $i, j = 1, \ldots, n$ *with* $i \neq j$, *where*

$$l_{ij}(\bar{x}) = \left(\frac{\partial f_i(\bar{x})}{\partial x_i}\right)^{-1} \frac{\partial f_i(\bar{x})}{\partial x_j} + \left(\frac{\partial f_j(\bar{x})}{\partial x_j}\right)^{-1} \frac{\partial f_j(\bar{x})}{\partial x_i},$$

then $\bar{x}$ *is globally asymptotically stable.*

Proof. Let $x = (x_1, \ldots, x_n) \in \mathcal{D}$, we will prove that the quadratic form

$$\begin{aligned} G(\bar{x}, x) &= x^T Hg(\bar{x})x \\ &= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial^2 g(\bar{x})}{\partial x_j \partial x_k} x_j x_k \right), \end{aligned} \tag{9}$$

is positive. Since

$$\frac{\partial^2 g(\bar{x})}{\partial x_j \partial x_k} = \frac{\partial^2 g(\bar{x})}{\partial x_k \partial x_j},$$

for $j, k = 1, \ldots, n$, then (9) is rewritten as:

$$\begin{aligned} G(\bar{x}, x) &= \sum_{k=1}^{n} \frac{\partial^2 g(\bar{x})}{\partial x_k^2} x_k^2 + 2\sum_{k=2}^{n} \frac{\partial^2 g(\bar{x})}{\partial x_1 \partial x_k} x_1 y_k + 2\sum_{k=3}^{n} \frac{\partial^2 g(\bar{x})}{\partial x_2 \partial x_k} x_2 y_k \\ &\quad + \cdots + 2\frac{\partial^2 g(\bar{x})}{\partial x_{n-1} \partial x_n} x_{n-1} x_n. \end{aligned} \tag{10}$$

Substituting (8) in (10) we have:

$$\begin{aligned} G(\bar{x}, x) &= -\sum_{k=1}^{n} 2\frac{a_k}{\bar{x}_k} \frac{\partial f_k(\bar{x})}{\partial x_k} x_k^2 - \sum_{k=2}^{n} 2\left(\frac{a_k}{\bar{x}_k} \frac{\partial f_k(\bar{x})}{\partial x_1} + \frac{a_1}{\bar{x}_1} \frac{\partial f_1(\bar{x})}{\partial x_k} \right) y_1 x_k \\ &\quad - \sum_{k=3}^{n} 2\left(\frac{a_k}{\bar{x}_k} \frac{\partial f_k(\bar{x})}{\partial x_2} + \frac{a_2}{\bar{x}_2} \frac{\partial f_2(\bar{x})}{\partial x_k} \right) x_2 x_k + \cdots + \\ &\quad -2\left(\frac{a_n}{\bar{x}_n} \frac{\partial f_n(\bar{x})}{\partial x_{n-1}} + \frac{a_{n-1}}{\bar{x}_{n-1}} \frac{\partial f_{n-1}(\bar{x})}{\partial x_n} \right) x_{n-1} x_n. \end{aligned} \tag{11}$$

Let

$$a_k = -\bar{x}_k \left[2\frac{\partial f_k(\bar{x})}{\partial x_k} \right]^{-1}, \; k = 1, \ldots, n. \tag{12}$$

Substituting (12) in (11) we have:

$$\begin{aligned}
G(\bar{x},x) &= \sum_{k=1}^{n} x_k^2 + \sum_{k=2}^{n}\left[\left(\frac{\partial f_k(\bar{x})}{\partial x_k}\right)^{-1}\frac{\partial f_k(\bar{x})}{\partial x_1} + \left(\frac{\partial f_1(\bar{x})}{\partial x_1}\right)^{-1}\frac{\partial f_1(\bar{x})}{\partial x_k}\right] x_1x_k \\
&\quad + \sum_{k=3}^{n}\left[\left(\frac{\partial f_k(\bar{x})}{\partial x_k}\right)^{-1}\frac{\partial f_k(\bar{x})}{\partial x_2} + \left(\frac{\partial f_2(\bar{x})}{\partial x_2}\right)^{-1}\frac{\partial f_2(\bar{x})}{\partial x_k}\right] x_2x_k \\
&\quad + \cdots + \left[\left(\frac{\partial f_n(\bar{x})}{\partial x_n}\right)^{-1}\frac{\partial f_n(\bar{x})}{\partial x_{n-1}} + \left(\frac{\partial f_{n-1}(\bar{x})}{\partial x_{n-1}}\right)^{-1}\frac{\partial f_{n-1}(\bar{x})}{\partial x_n}\right] x_{n-1}x_n \\
&> \sum_{k=1}^{n} x_k^2 + \sum_{k=2}^{n} 2x_1x_k + \sum_{k=3}^{n} 2x_2x_k + \cdots + 2x_{n-1}x_n \\
&= (x_1 + x_2 + \cdots + x_n)^2 \\
&> 0. \qquad (13)
\end{aligned}$$

In consequence, from Corollary 4 we conclude that $\bar{x}$ is globally asymptotically stable in $\mathcal{D}$.

4 Application of main result

In this section we will apply the Theorem 5 to prove the asymptotic stability of nontrivial equilibrium of the nonlinear system

$$\frac{dx_j}{dt} = \alpha_j x_j(1-x_j) - \sigma_j \prod_{i=1}^{n} x_i, \quad j = 1,2,\ldots,n, \qquad (14)$$

where $0 < \alpha_j < 1$ and $0 < \sigma_j < 1$ for $j = 1,2,\ldots,n$. Our set of interest is:

$$\mathcal{D}_1 = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1, i, j = 1,2,\ldots,n\}. \qquad (15)$$

The following lemma ensures that all solutions of (14) starting in $\mathcal{D}_1$ remain there for all $t \geq 0$.

Lemma 6. *The set $\mathcal{D}_1$ defined in (15) is positively invariant for the solutions of the system (14).*

Proof. Let $x = (x_1^0, x_2^0, \ldots, x_n^0)$ be given. If there is $1 \leq j \leq n$ such that $x_j^0 = 0$, then we see directly from the unique and existent result that $x_j(t) \equiv 0$ for all $t \geq 0$, and so for $k \neq j$ such that $x_k^0 \neq 0$, we have that $x_k(t)$ satisfies the logic differential equation:

$$\frac{dx_k}{dt} = \alpha_k x_k(1-x_k),$$

for which we know that $0 \leq x_k(t) \leq 1$. In other words, if there is $1 \leq j \leq n$ such that $x_j^0 = 0$, we have that $0 \leq x_k(t) \leq 1$ for $1 \leq k \leq n$. Now, we assume that $x = (x_1^0, x_2^0, \ldots, x_n^0) \in \mathcal{D}_1$ is such that $x_j^0 \neq 0$ for any $1 \leq j \leq n$. In this case, we know that $x_j(t)$ for any $t \geq 0$ and $0 \leq j \leq n$. Then, from (14) we obtain:

$$\frac{dx_j}{dt} \leq \alpha_j x_j(1-x_j), \quad j = 1,2,\ldots,n$$

or equivalently

$$-x_j^{-2}\frac{dx_j}{dt}+\alpha_j x_j^{-1}\geq \alpha_j, \quad j=1,2,\ldots,n. \tag{16}$$

Let $z=x_j^{-1}$, then $dz/dt=-x_j^{-2}dx_j/dt$. Substituting z and dz/dt in (16) we have

$$\frac{dz}{dt}+\alpha_j z\geq \alpha_j.$$

Multiplying the above inequality by $e^{\alpha_j t}$ we obtain:

$$\frac{d\left(e^{\alpha_j t}z\right)}{dt}\geq \alpha_j e^{\alpha_j t}. \tag{17}$$

Integrating the inequality (17) between 0 and t we have:

$$z(t)\geq 1+(z(0)-1)e^{-\alpha_j t}. \tag{18}$$

Substituting $z=x_j^{-1}$ in (18) we obtain:

$$x_j(t)\leq \frac{1}{1+\left[x_j^{-1}(0)-1\right]e^{-\alpha_j t}}.$$

Therefore, we conclude that:

$$0\leq x_j(t)\leq 1 \text{ for all } t\geq 0.$$

meaning $x=(x_1^0,x_2^0,\ldots,x_n^0)\in\mathcal{D}_1$ as desired.

The next proposition summarizes existent results of the equilibrium solutions of (14).

Proposition 2. *The system (14) has at least* $2^{n+1}-1$ *equilibrium solution in* $\mathcal{D}_1$.

Proof. The equilibrium solutions of (14) are given by the solutions of the algebraic system

$$\alpha_j x_j(1-x_j)-\sigma_j\prod_{i=1}^{n}x_i=0, \quad j=1,2,\ldots,n. \tag{19}$$

Observe that in the following cases, a)$x_j=0$, b)$x_j=1$ and $x_k=0$ for $j\neq k$, the equations (19) are satisfied, which implies the existence of 2^n-1 equilibrium of the form $x_0=(p_1,\ldots,p_n)$ where $p_j=0$ or $p_j=1$. On the other hand, from (19) we obtain:

$$\frac{\alpha_j}{\sigma_j}x_j(1-x_j)=k, \quad j=1,2,\ldots,n, \tag{20}$$

where $k=\prod_{i=1}^{n}x_i$. The solutions of (20) are:

$$x_j=\frac{1\pm\sqrt{1-4k\sigma_j/\alpha_j}}{2}, \quad j=1,2,\ldots,n.$$

The above implies that $x_j>0$ if and only if $0<k<\alpha_n/4\sigma_n$. Therefore, there are at least two equilibriums in $int\,(\mathcal{D}_1)$. This completes the proof.

The following proposition summarizes stability results of the equilibrium of (14).

Proposition 3. *Suppose that the system (14) has an interior steady state* $\bar{x} \in \mathcal{D}_2 \subset \mathcal{D}_1$ *where*

$$\mathcal{D}_2 = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1, 0 \leq x_i + x_j, \leq 1 \; i, j = 1, 2, \ldots, n\}.$$

then this steady state is globally asymptotically stable on the interior set of $\mathcal{D}_1$.

Proof. From (14) we conclude that:

$$f_j(x) = \alpha_j x_j(1 - x_j) - \sigma_j \prod_{k=1}^{n} x_k, \quad j = 1, 2, \ldots, n,$$

which implies that

$$\frac{\partial f_j(\bar{x})}{\partial \bar{x}_j} = \alpha_j(1 - \bar{x}_j) - \alpha_j \bar{x}_j - \sigma_j \prod_{k=1, k\neq j}^{n} \bar{x}_k, \quad j = 1, 2, \ldots, n. \tag{21}$$

From equilibrium equations we have:

$$\alpha_j(1 - \bar{x}_j) - \sigma_j \prod_{k=1, k\neq j}^{n} \bar{x}_k = 0, \quad j = 1, 2, \ldots, n. \tag{22}$$

Therefore, substituting (22) in (21) we verify the first hypothesis of Theorem 5, that is

$$\frac{\partial f_j(\bar{x})}{\partial \bar{x}_j} = -\alpha_j \bar{x}_j < 0, \quad j = 1, 2, \ldots, n.$$

On the other hand,

$$\begin{aligned} l_{ij}(\bar{x}) &= \left(\frac{\partial f_i(\bar{x})}{\partial x_i}\right)^{-1} \frac{\partial f_i(\bar{x})}{\partial x_j} + \left(\frac{\partial f_j(\bar{x})}{\partial x_j}\right)^{-1} \frac{\partial f_j(\bar{x})}{\partial x_i} \\ &= (-\alpha_i \bar{x}_i)^{-1} \left(-\sigma_i \prod_{k=1, k\neq j}^{n} \bar{x}_k\right) + (-\alpha_j \bar{x}_j)^{-1} \left(-\sigma_j \prod_{k=1, k\neq i}^{n} \bar{x}_k\right) \\ &= \left(\frac{\sigma_i}{\alpha_i} + \frac{\sigma_j}{\alpha_j}\right) \prod_{k=1, k\neq i, k\neq j}^{n} \bar{x}_k. \end{aligned} \tag{23}$$

From (22) we have:

$$\frac{\sigma_j}{\alpha_j} = \frac{1 - \bar{x}_j}{\prod_{k=1, k\neq j}^{n} \bar{x}_k}, \quad j = 1, 2, \ldots, n. \tag{24}$$

Substituting (24) in (23) we obtain:

$$l_{ij}(\bar{x}) = \frac{1 - \bar{x}_i}{\bar{x}_j} + \frac{1 - \bar{x}_j}{\bar{x}_i}, \quad \text{for } i \neq j.$$

From hypothesis $\bar{x} \in \mathcal{D}_2$, results that $0 < \bar{x}_i + \bar{x}_j < 1$ which implies $(\bar{x}_i + \bar{x}_j)^2 < \bar{x}_i + \bar{x}_j$, or equivalently $(1 - \bar{x}_i)\bar{x}_i + (1 - \bar{x}_j)\bar{x}_j > 2\bar{x}_i\bar{x}_j$. The above implies that the second hypothesis of Theorem 5 is satisfied. That is $l_{ij} > 2$. Therefore $\bar{x}$ is globally asymptotically stable on interior set of $\mathcal{D}_1$.

4.1 Numerical solutions

One of the possible applications for system (14) when $n = 3$ could be the competition among three species with logistic growth. The simulation of Figure 1 was made with the following data: $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\alpha_3 = 0.15$, $\sigma_1 = 0.08$, $\sigma_2 = 0.15$ and $\sigma_3 = 0.14$. In this case the solutions of (14) tend to the coexistent equilibrium $P_1 = (0.68, 0.72, 0.53)$ which agrees with the theoretical results.

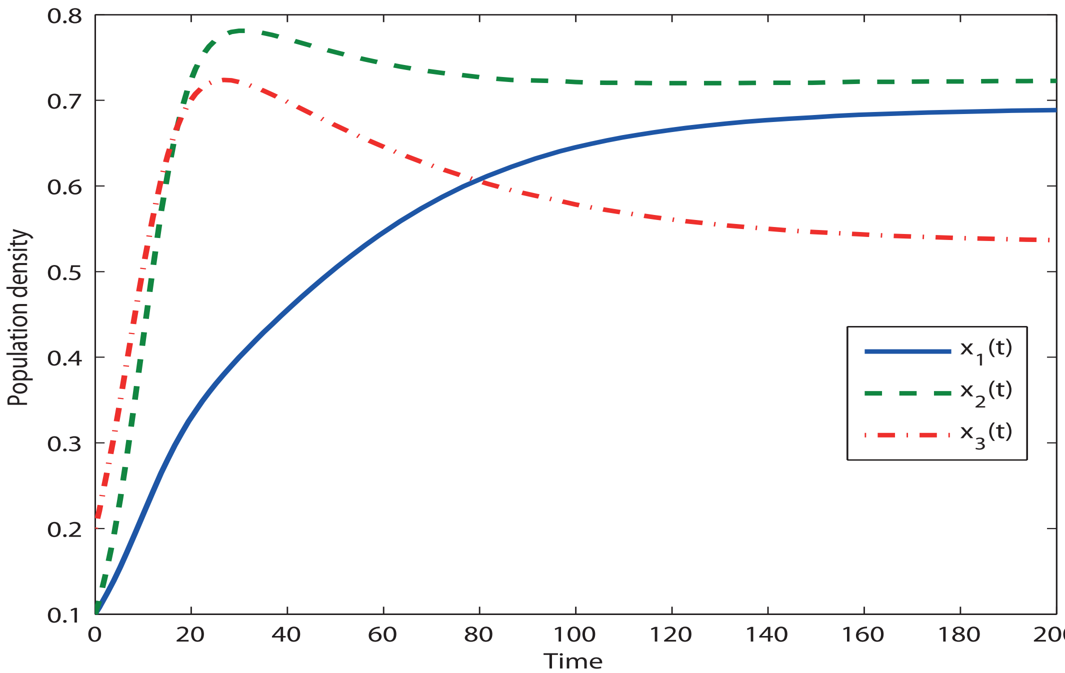

Figure 1: *Graphs of the component solutions x_1, x_2 and x_3 of (14) for $n = 3$. In this case, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\alpha_3 = 0.15$, $\sigma_1 = 0.08$, $\sigma_2 = 0.15$, $\sigma_3 = 0.14$.*

5 Conclusión

In certain areas of applied mathematics such as Biomathematics, the qualitative analysis of the solutions of dynamical systems defined by ordinary differential equations is fundamental to understand problems in biology (Ibargüen et al. [11]). In this sense, the DML is very practical and widely used to analyze the stability of dynamical systems. In this article we use the DML to establish easier conditions to verify the assurance of global asymptotic stability of the equilibrium solutions of some dynamical systems. The fact that these conditions are defined in terms of $\partial f_i(\bar{x})/\partial x_i$ for $i = 1, \ldots, n$, suggest the possibility that the stability test (Theorem 5) can be used to numerically verify asymptotic stability.

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