Resolviendo la ecuación de Pell con convergentes de orden inferior

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Resumen

El método de fracciones continuas provee la menor solución entera no trivial de la ecuación de Pell. Dicha solución se expresa en términos del convergente de la fracción continua de la raíz cuadrada de un número entero no cuadrado al final de su primer o segundo período. En este trabajo, se obtiene que este convergente a su vez se expresa en términos de otro convergente de menor orden de la misma fracción continua.

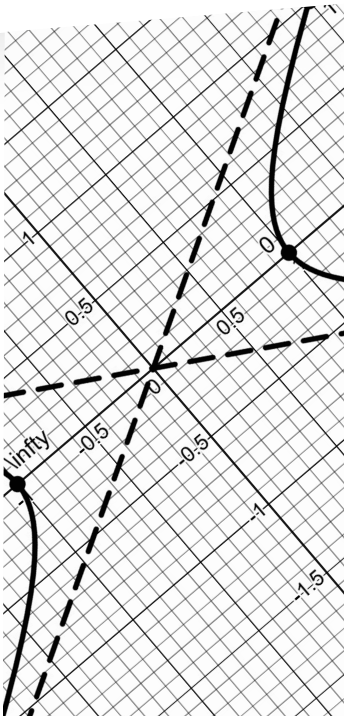
Palabras clave: ecuación de Pell, fracción continua, convergentes, parametrización racional, hipérbola.

Solving Pell's Equation by Lower - Order Convergents

Abstract

The method of continued fractions provides the smallest non-trivial integer solution of Pell's equation. This solution comes from the convergent of the continued fraction of the square root of an integer number that is not a square at the end of its first or its second period. In this work, it is obtained that this convergent is expressed in terms of a lower-order convergent of the same continued fraction.

Keywords: Pell's equation, continued fraction, convergents, rational parametrization, hyperbola.



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1. Introduction

The resolution of generalized Pell's equation

$$ny^2 - x^2 = m, \qquad (1)$$

where m and n are integers and n is not a square, makes use of algorithms that involve many calculations. Due to the development of computers, this subject has become recently a topic of interest. It is known that all the integer solutions (x_i, y_i) for Equation (1) are derived from the recursion formula (see [1])

$$(x_{i+1}, y_{i+1}) = (\alpha x_i + n\beta y_i, \beta x_i + \alpha y_i), \qquad (2)$$

where (x_0, y_0) is the smallest integer solution of Equation (1) and (α, β) is the smallest non-trivial integer solution of Pell's equation

$$ny^2 - x^2 = -1. \qquad (3)$$

Integer solutions of Pell's equation (3) are determined by the convergents of the continued fraction of the irrational number $\sqrt{n}$, all these elements having interesting properties (Sections 3 and 4). In fact, the coordinates (k_s, q_s) of the smallest non-trivial integer solution of Pell's equation (3) are determined by a specific convergent $\frac{k_s}{q_s}$ of the mentioned continued fraction (Theorem 5.1).

However, the way how convergents of continued fractions are defined requires that all the previous ones be calculated to reach the solution.

In this work, an oriented parametrization for the rational points of the hyperbola $H: ny^2 - x^2 = -1$ is constructed in Section 2, and is used to relate the convergent $\frac{k_s}{q_s}$ that solves Pell's equation (3) with a convergent $\frac{k_*}{q_*}$ of lower order. We use classic properties of continued fractions and convergents to establish the main result Theorem 5.2 in Section 5, which in turn derives in expressions for each coordinate of the smallest non-trivial integer solution of Pell's equation (3) in terms of the same lower-order convergents in Proposition 5.3.

2. Rational Numbers as Rational Points of a Hyperbola

For real numbers $\{x, y, k, q, n\}$, there is an identity

$$\begin{aligned}(nq^2 - k^2)(ny^2 - x^2) &= n^2q^2y^2 - n(q^2x^2 + k^2y^2) + k^2x^2 \\ &= -n(q^2x^2 + 2kqxy + k^2y^2) + n^2q^2y^2 + 2nkqxy + k^2x^2 \\ &= -n(qx + ky)^2 + (nqy + kx)^2 = -[n(qx + ky)^2 - (nqy + kx)^2].\end{aligned}$$

When taking $(k, q) = (x, y)$, Brahmagupta's identity takes the form

$$n(2kq)^2 - (nq^2 + k^2)^2 = -(nq^2 - k^2)^2. \tag{4}$$

Then

$$n\left(\frac{2kq}{nq^2 - k^2}\right)^2 - \left(\frac{nq^2 + k^2}{nq^2 - k^2}\right)^2 = -1, \tag{5}$$

and the ordered pair

$$(x, y) = \left(\frac{nq^2 + k^2}{nq^2 - k^2}, \frac{2kq}{nq^2 - k^2}\right) \tag{6}$$

is a rational solution of Pell's equation (3) when k and q are integers and n is a natural number that is not a square. It is found an inclusion of the set Q of rational numbers in the hyperbola $H: ny^2 - x^2 = -1$ as described in the following statement.

Proposition 2.1. *Let be a natural number that is not a square. There is a bijection between the set $Q \cup \{\infty\}$ of extended rational numbers and the points of the hyperbola $H: ny^2 - x^2 = -1$ that have both entries rational.*

Proof. The function $f: Q \cup \{\infty\} \to H$ is defined as

$$f(\alpha) = \{\left(\frac{nq^2 + k^2}{nq^2 - k^2}, \frac{2kq}{nq^2 - k^2}\right), \quad if\ \alpha = \frac{k}{q} \in Q; (-1, 0), \quad if\ \alpha = \infty. \tag{7}$$

If we write $\frac{k}{q} = \alpha$, then

$$f(\alpha) = \left(\frac{n + \alpha^2}{n - \alpha^2}, \frac{2\alpha}{n - \alpha^2}\right). \tag{8}$$

To prove that f is injective, let us take rational numbers α and β such that $f(\alpha) = f(\beta)$. Then

$$\frac{n+\alpha^2}{n-\alpha^2} = \frac{n+\beta^2}{n-\beta^2}, \tag{9}$$

and

$$\frac{2\alpha}{n-\alpha^2} = \frac{2\beta}{n-\beta^2}. \tag{10}$$

Equation (9) leads to $\alpha^2 = \beta^2$ and then Equation (10) leads to $\alpha = \beta$. Then the function f is injective.

To prove surjection, let us consider a point $(x, y) \in H$ with both entries rational. Then x and y are rational numbers such that $ny^2 - x^2 = -1$. Let us consider the extended rational number

$$\frac{k}{q} = \frac{ny}{x+1}. \tag{11}$$

Then

$$\left(\frac{k}{q}\right)^2 = \frac{n^2y^2}{(x+1)^2} = \frac{n(x-1)}{x+1}, \tag{12}$$

and we reach that

$$\frac{n+\left(\frac{k}{q}\right)^2}{n-\left(\frac{k}{q}\right)^2} = x, \tag{13}$$

and

$$\frac{2\cdot\frac{k}{q}}{n-\left(\frac{k}{q}\right)^2} = y. \tag{14}$$

Therefore,

$$(x,y) = f\left(\frac{ny}{x+1}\right), \tag{15}$$

and there is a bijection between the set $Q \cup \{\infty\}$ of extended rational numbers and the points of the hyperbola $H{:}ny^2 - x^2 = -1$ with both entries rational. ■

The bijection $f: Q \cup \{\infty\} \to H \cap (Q \times Q)$ provides an orientation for the hyperbola $H{:}ny^2 - x^2 = -1$ induced by the usual ordering of rational numbers as displayed in Figure 1: starting at the vertex $(-1,0) \in H$ corresponding to $\frac{k}{q} = \infty$, the rational point $f\left(\frac{k}{q}\right)$ gets closer to the line $\sqrt{n}y + x = 0$ in the second quadrant of the plane while $\frac{k}{q}$ is increasing and keeping lower than $-\sqrt{n}$. Once $\frac{k}{q} > -\sqrt{n}$, the rational point $f\left(\frac{k}{q}\right)$ jumps to the fourth quadrant arbitrarily close to the same asymptote and begins to reach the vertex (1,0) which corresponds to $\frac{k}{q} = 0$. When $\frac{k}{q}$ begins to be positive, the rational point $f\left(\frac{k}{q}\right)$ continues its way on the first quadrant getting closer to the line $\sqrt{n}y - x = 0$ while $\frac{k}{q} < \sqrt{n}$. Once $\frac{k}{q} > \sqrt{n}$, the rational point $f\left(\frac{k}{q}\right)$ jumps to the third quadrant arbitrarily close to the same asymptote and goes back to the vertex $(-1,0)$ as $\frac{k}{q} \to \infty$.

Let us recall a basic property of hyperbolas.

Remark 2.2. If (x, y) and (ax, ay) are points of the hyperbola $ny^2 - x^2 = m$, then $a^2 = 1$.

3. Continued Fractions of Square Roots of Natural Numbers

To find integer solutions of Pell's equation (3), it is sufficient to find integers k and q such that $nq^2 - k^2$ divides both $nq^2 + k^2$ and $2kq$ according to Proposition 2.1. Therefore, the pair (k, q) must be chosen so that $nq^2 - k^2$ be sufficiently small. A method for obtaining such suitable values for k and q is by approximating the irrational number $\sqrt{n}$ via continued fractions. A continued fraction is an expression of the form

$$[a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}, \tag{16}$$

where each a_i is an integer number and $a_i > 0$ when $i > 0$. For example, let us calculate the continued fraction for $\alpha = \sqrt{13}$: it is known that $3 < \alpha < 4$; moreover, $\alpha^2 - 3^2 = 4$. Then $\alpha - 3 = \frac{4}{3+\alpha}$ and it follows that

$$\alpha = 3+\cfrac{1}{\cfrac{3+\alpha}{4}} = 3+\cfrac{1}{1+\cfrac{\alpha-1}{4}} = 3+\cfrac{1}{1+\cfrac{1}{\cfrac{4}{\alpha-1}}} = 3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{5-\alpha}{\alpha-1}}}$$

$$= 3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{\alpha-1}{5-\alpha}}}} = 3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2\alpha-6}{5-\alpha}}}} = 3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{5-\alpha}{2\alpha-6}}}}}$$

$$= 3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{11-3\alpha}{2\alpha-6}}}}} = 3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{2\alpha-6}{11-3\alpha}}}}}}$$

$$= 3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{20\alpha-72}{11-3\alpha}}}}}}.$$

For the case of the square root of a positive integer number, its continued fraction is periodic with the number $2\lfloor\sqrt{n}\rfloor = 2a_0$ being the last element in the period. Furthermore, the remaining elements in the period have a symmetric behavior as observed in [2]: if $\sqrt{n} = [a_0; \overline{a_1, \dots, a_{p-1}, 2a_0}]$, then, $a_{p-i} = a_i$ for every $0 < i < p$. It follows that $\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}]$ is a continued fraction with period of length $p = 5$. We could test the pair $(k, q) = (72, 20)$ coming from the polynomial expression $20\alpha - 72$ in the continued fraction of $\sqrt{13}$, obtaining the value $13 \cdot 20^2 - 72^2 = 16$, and so the values for x and y would be

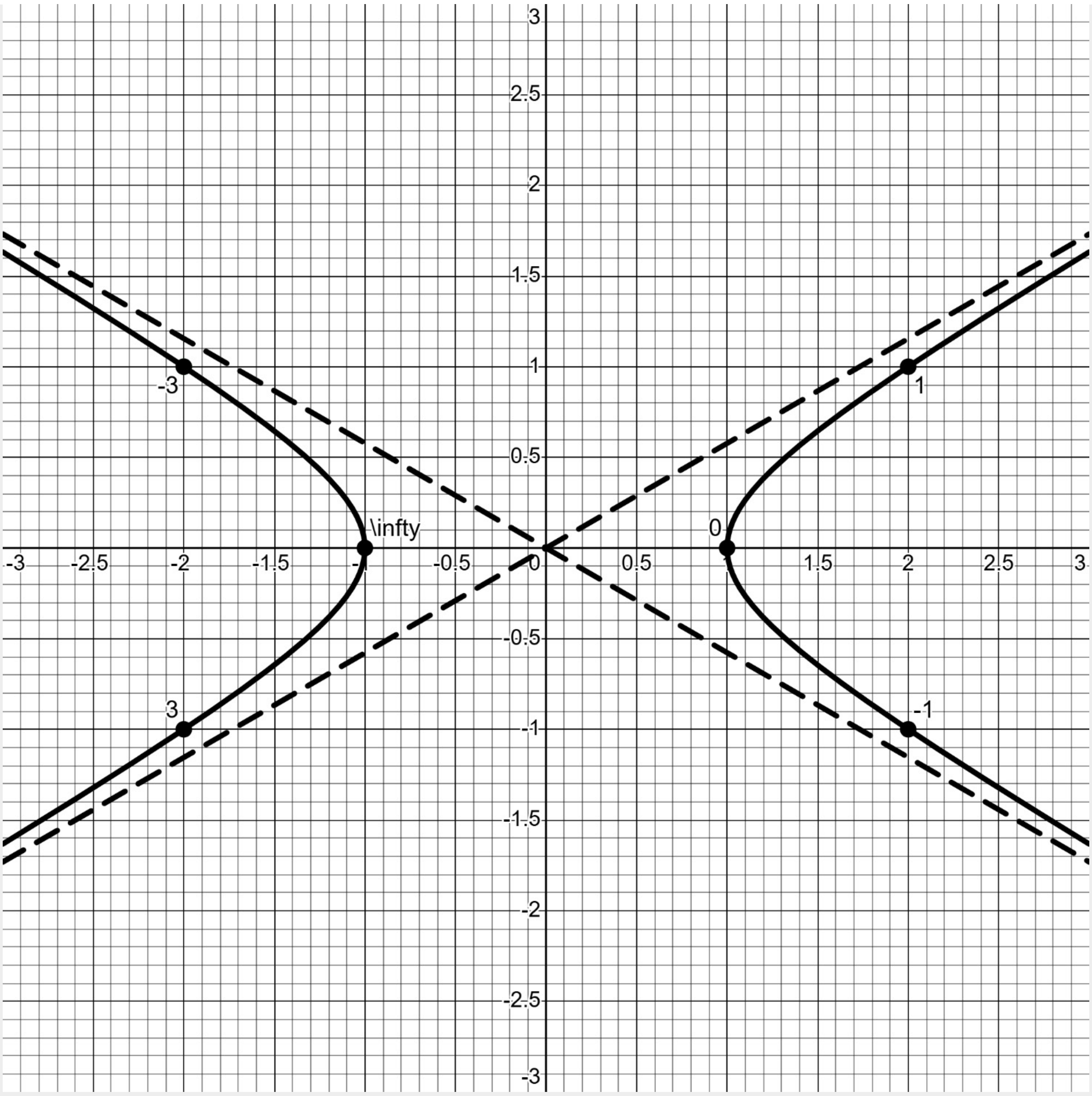

Figure 1. Rational points of the hyperbola $3y^2 - x^2 = -1$ regarded as extended rational numbers.

$$x = \frac{13 \cdot 20^2 + 72^2}{13 \cdot 20^2 - 72^2} = \frac{10384}{16} = 649, \tag{17}$$

and

$$y = \frac{2 \cdot 72 \cdot 20}{13 \cdot 20^2 - 72^2} = \frac{2880}{16} = 180, \tag{18}$$

which is the smallest non-trivial integer solution of Pell's equation $13y^2 - x^2 = -1$.

4. Review on Convergents

For any continued fraction $[a_0; a_1, a_2, \dots]$, its -*th convergent* $\frac{k_i}{q_i}$ is defined inductively by making, $(k_{-1}, q_{-1}) = (1, 0)$, $(k_0, q_0) = (a_0, 1)$, and $(k_i, q_i) = (a_i k_{i-1} + k_{i-2}, a_i q_{i-1} + q_{i-2})$.

The i-th convergent $\frac{k_i}{q_i}$ can be calculated alternatively as the finite continued fraction

$$\frac{k_i}{q_i} = [a_0; a_1, \dots, a_i]. \tag{19}$$

In fact, it is proved (see [2]) that

$$[a_0; a_1, \dots, a_{i-1}, \alpha] = \frac{\alpha k_{i-1} + k_{i-2}}{\alpha q_{i-1} + q_{i-2}}, \tag{20}$$

for every positive real number a.

Other properties of interest with respect to convergents are the *associativity*

$$[a_0; a_1, \dots, a_{i-1}, a_i, \alpha] = \left[a_0; a_1, \dots, a_{i-1}, [a_i; \alpha]\right], \tag{21}$$

the *reduction*

$$[a_0; a_1, \dots, a_{i-1}, \alpha] = \left[a_0; a_1, \dots, a_{i-1} + \frac{1}{\alpha}\right], \tag{22}$$

the *inversion*

$$\frac{q_i}{q_{i-1}} = [a_i; a_{i-1}, \dots, a_1], \tag{23}$$

the *difference*

$$k_{i-1} q_i - k_i q_{i-1} = (-1)^i, \tag{24}$$

and the *monotonicity*

$$\frac{k_0}{q_0} < \frac{k_2}{q_2} < \cdots < \frac{k_{2i}}{q_{2i}} < \cdots < [a_0; a_1, a_2, \dots] < \cdots < \frac{k_{2i+1}}{q_{2i+1}} < \cdots < \frac{k_3}{q_3} < \frac{k_1}{q_1}. \tag{25}$$

We are proving inversion property (23) as follows: if i = 1, then q_{-1} = 0, q_0 = 1, and

$$q_1 = a_1 q_0 + q_{-1} = a_1, \tag{26}$$

following that

$$\frac{q_1}{q_0} = a_1 = [a_1]. \tag{27}$$

Now, let us suppose that inversion property (*23*) holds for i and let us calculate

$$\frac{q_{i+1}}{q_i} = \frac{a_{i+1}q_i + q_{i-1}}{q_i} = a_{i+1} + \frac{q_{i-1}}{q_i} = \left[a_{i+1}; \frac{q_i}{q_{i-1}}\right] = [a_{i+1}; [a_i; \dots, a_1]]$$
$$= [a_{i+1}; a_i, \dots, a_1].$$

Then inversion property (23) holds for i + 1 and it holds for every positive integer i.

We use monotonicity property (25) to locate the image of the convergents via the parametrization f from Proposition 2.1.v

Lemma 4.1. *Let n be a natural number that is not a square and let us consider the continued fraction of* $\sqrt{n}$ *with convergents* $\frac{k_i}{q_i}$*. If is odd, then the point* $f\left(\frac{k_i}{q_i}\right)$ *belongs to the third quadrant. If is even, then the point* $f\left(\frac{k_i}{q_i}\right)$ *belongs to the first quadrant.*

Proof. If i is odd, then $\frac{k_i}{q_i} > \sqrt{n}$. Then $nq_i^2 - k_i^2 < 0$ and the rational point

$$f\left(\frac{k_i}{q_i}\right) = \left(\frac{nq_i^2 + k_i^2}{nq_i^2 - k_i^2}, \frac{2k_i q_i}{nq_i^2 - k_i^2}\right)$$

has both entries negative.

If i is even, then $0 < \frac{k_i}{q_i} < \sqrt{n}$. Then $nq_i^2 - k_i^2 > 0$ and the rational point

$$f\left(\frac{k_i}{q_i}\right) = \left(\frac{nq_i^2 + k_i^2}{nq_i^2 - k_i^2}, \frac{2k_i q_i}{nq_i^2 - k_i^2}\right)$$

has both entries positive. ■

5. The Result

The smallest non-trivial integer solution of Pell's equation (3) has been calculated in terms of the convergents of the continued fraction of $\sqrt{n}$ as proved by C. D. Olds in [2]:

Theorem 5.1. *Let n be a natural number that is not a square and let us consider the continued fraction* $\sqrt{n} = [a_0; \overline{a_1, \dots, a_{p-1}, 2a_0}]$ *with period of length* p *and its convergents* $\frac{k_i}{q_i}$. *If* p *is even, then the pair* $(k_s, q_s) = (k_{p-1}, q_{p-1})$ *is the first non-trivial integer solution of* $ny^2 - x^2 = -1$. *If* p *is odd, then the pair* $(k_s, q_s) = (k_{2p-1}, q_{2p-1})$ *is the first non-trivial integer solution of* $ny^2 - x^2 = -1$. *Also, the equality* $nq_{p-1}^2 - k_{p-1}^2 = (-1)^{p-1}$ *holds for any length* p.

Let us calculate the first non-trivial integer solution of Pell's equation $13y^2 - x^2 = -1$ by using Theorem 5.1: since the length p of the period of the continued fraction $\sqrt{13} = [3; \overline{1,1,1,1,6}]$ is $p = 5$, Theorem 5.1 commands us to calculate the pair $(k_{2\cdot5-1}, q_{2\cdot5-1}) = (k_9, q_9)$. We proceed by making:

$$\begin{aligned}
(k_1, q_1) &= (1\cdot3+1, 1\cdot1+0) = (4,1)\\
(k_2, q_2) &= (1\cdot4+3, 1\cdot1+1) = (7,2)\\
(k_3, q_3) &= (1\cdot7+4, 1\cdot2+1) = (11,3)\\
(k_4, q_4) &= (1\cdot11+7, 1\cdot3+2) = (18,5)\\
(k_5, q_5) &= (6\cdot18+11, 6\cdot5+3) = (119,33)\\
(k_6, q_6) &= (1\cdot119+18, 1\cdot33+5) = (137,38)\\
(k_7, q_7) &= (1\cdot137+119, 1\cdot38+33) = (256,71)\\
(k_8, q_8) &= (1\cdot256+137, 1\cdot71+38) = (393,109)\\
(k_9, q_9) &= (1\cdot393+256, 1\cdot109+71) = (649,180).
\end{aligned}$$

The pair $(k_9, q_9) = (649, 180)$ is the smallest non-trivial integer solution of Pell's equation $13y^2 - x^2 = -1$. However, we may pay attention to the convergent $\frac{k_4}{q_4} = \frac{18}{5}$ and we calculate

$$f\left(\frac{k_4}{q_4}\right) = f\left(\frac{18}{5}\right) = \left(\frac{13\cdot5^2+18^2}{13\cdot5^2-18^2}, \frac{2\cdot18\cdot5}{13\cdot5^2-18^2}\right) = (649,180) = (k_9, q_9).$$

For the even case, let us consider the continued fraction

$$\sqrt{1986} = [44; \overline{1,1,3,2,1,2,5,1,1,3,44,3,1,1,5,2,1,2,3,1,1,88}] \tag{28}$$

which has a period of length $p = 22$. According to Theorem 5.1, the first non-trivial integer solution of Pell's equation $1986y^2 - x^2 = -1$ is $(k_{22-1}, q_{22-1}) = (k_{21}, q_{21}) = (13209364625, 296409628)$, but we can also calculate

$$f^{-1}(k_{21}, q_{21}) = f^{-1}(13209364625, 296409628) = \frac{1986\cdot296409628}{13209364625+1} = \frac{114932}{2579} = \frac{k_{10}}{q_{10}}.$$

The bijection $f: Q \cup \{\infty\} \to H \cap (Q \times Q)$ from Proposition 2.1 establishes a connection between the smallest non-trivial integer solution of Pell's equation (3) and a convergent of lower order. We prove the relation in the following statement:

Theorem 5.2. *Let n be a natural number that is not a square and let us consider the continued fraction*

$$\sqrt{n} = [a_0; \overline{a_1, \dots, a_{p-1}, 2a_0}] \tag{29}$$

with period of length p and its convergents $\frac{k_i}{q_i}$. *If p is even, then*

$$\frac{nq_{\frac{p}{2}-1}^2 + k_{\frac{p}{2}-1}^2}{2k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}} = \frac{k_{p-1}}{q_{p-1}}. \tag{30}$$

If p is odd, then

$$\frac{nq_{p-1}^2 + k_{p-1}^2}{2k_{p-1}q_{p-1}} = \frac{k_{2p-1}}{q_{2p-1}}. \tag{31}$$

Proof. Let us suppose that the length p of the period is odd and let us observe that

$$\frac{k_{p-1}}{q_{p-1}} = [a_0; a_1, \dots, a_{p-1}] = [a_0; a_{p-1}, \dots, a_1] = [a_0; [a_{p-1}; \dots, a_1]] = \left[a_0; \frac{q_{p-1}}{q_{p-2}}\right] = a_0 + \frac{q_{p-2}}{q_{p-1}}.$$

Then $k_{p-1} = a_0 q_{p-1} + q_{p-2}$, leading to $2k_{p-1}q_{p-1} = a_0 q_{p-1}^2 + k_{p-1}q_{p-1} + q_{p-2}q_{p-1}$ and

$$\frac{1}{2k_{p-1}q_{p-1}} = \frac{1}{a_0 q_{p-1}^2 + k_{p-1}q_{p-1} + q_{p-2}q_{p-1}}. \tag{32}$$

Since p is odd, it follows that $(-1)^{p-1} = 1$ and the pair (k_{p-1}, q_{p-1}) is a solution of the equation $ny^2 - x^2 = 1$ by Theorem 5.1. Then

$$\begin{aligned}
\frac{nq_{p-1}^2 - k_{p-1}^2}{2k_{p-1}q_{p-1}} &= \frac{(-1)^{p-1}}{a_0 q_{p-1}^2 + k_{p-1}q_{p-1} + q_{p-2}q_{p-1}} = \frac{k_{p-2}q_{p-1} - k_{p-1}q_{p-2}}{a_0 q_{p-1}^2 + k_{p-1}q_{p-1} + q_{p-2}q_{p-1}} \\
&= \frac{a_0 k_{p-1}q_{p-1}^2 + k_{p-1}^2 q_{p-1} + k_{p-2}q_{p-1}^2 - a_0 k_{p-1}q_{p-1}^2 - k_{p-1}^2 q_{p-1} - k_{p-1}q_{p-2}q_{p-1}}{q_{p-1}(a_0 q_{p-1}^2 + k_{p-1}q_{p-1} + q_{p-2}q_{p-1})} \\
&= \frac{q_{p-1}(a_0 k_{p-1}q_{p-1} + k_{p-1}^2 + k_{p-2}q_{p-1}) - k_{p-1}(a_0 q_{p-1}^2 + k_{p-1}q_{p-1} + q_{p-2}q_{p-1})}{q_{p-1}(a_0 q_{p-1}^2 + k_{p-1}q_{p-1} + q_{p-2}q_{p-1})} \\
&= \frac{a_0 k_{p-1}q_{p-1} + k_{p-1}^2 + k_{p-2}q_{p-1}}{a_0 q_{p-1}^2 + k_{p-1}q_{p-1} + q_{p-2}q_{p-1}} - \frac{k_{p-1}}{q_{p-1}} = \frac{\left(a_0 + \frac{k_{p-1}}{q_{p-1}}\right)k_{p-1} + k_{p-2}}{\left(a_0 + \frac{k_{p-1}}{q_{p-1}}\right)q_{p-1} + q_{p-2}} - \frac{k_{p-1}}{q_{p-1}} \\
&= \left[a_0; a_1, \dots, a_{p-1}, a_0 + \frac{k_{p-1}}{q_{p-1}}\right] - \frac{k_{p-1}}{q_{p-1}} = [a_0; a_1, \dots, a_{p-1}, a_0 + [a_0; a_1, \dots, a_{p-1}]] - \frac{k_{p-1}}{q_{p-1}} \\
&= [a_0; a_1, \dots, a_{p-1}, [2a_0; a_1, \dots, a_{p-1}]] - \frac{k_{p-1}}{q_{p-1}} = [a_0; a_1, \dots, a_{p-1}, 2a_0, a_1, \dots, a_{p-1}] - \frac{k_{p-1}}{q_{p-1}} \\
&= \frac{k_{2p-1}}{q_{2p-1}} - \frac{k_{p-1}}{q_{p-1}}.
\end{aligned}$$

It follows that

$$\frac{nq_{p-1}^2 - k_{p-1}^2}{2k_{p-1}q_{p-1}} + \frac{k_{p-1}}{q_{p-1}} = \frac{k_{2p-1}}{q_{2p-1}}, \tag{33}$$

and it is concluded that

$$\frac{nq_{p-1}^2 + k_{p-1}^2}{2k_{p-1}q_{p-1}} = \frac{k_{2p-1}}{q_{2p-1}}, \tag{34}$$

if the length p is odd.

If the length p is even, P. J. Rippon and H. Taylor proved in [3] that there is an identity

$$n = \frac{k_{\frac{p}{2}-1}\left(k_{\frac{p}{2}} + k_{\frac{p}{2}-2}\right)}{q_{\frac{p}{2}-1}\left(q_{\frac{p}{2}} + q_{\frac{p}{2}-2}\right)}. \tag{35}$$

Then

$$n = \frac{k_{\frac{p}{2}-1}k_{\frac{p}{2}} + k_{\frac{p}{2}-2}k_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}q_{\frac{p}{2}} + q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}} = \frac{k_{\frac{p}{2}-1}\left(a_{\frac{p}{2}}k_{\frac{p}{2}-1} + k_{\frac{p}{2}-2}\right) + k_{\frac{p}{2}-2}k_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}\left(a_{\frac{p}{2}}q_{\frac{p}{2}-1} + q_{\frac{p}{2}-2}\right) + q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}},$$

and we obtain

$$n\left(a_{\frac{p}{2}}q_{\frac{p}{2}-1}^2 + 2q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}\right) = a_{\frac{p}{2}}k_{\frac{p}{2}-1}^2 + 2k_{\frac{p}{2}-2}k_{\frac{p}{2}-1}. \tag{36}$$

Then

$$na_{\frac{p}{2}}q_{\frac{p}{2}-1}^2 + 2nq_{\frac{p}{2}-2}q_{\frac{p}{2}-1} - a_{\frac{p}{2}}k_{\frac{p}{2}-1}^2 = 2k_{\frac{p}{2}-2}k_{\frac{p}{2}-1}. \tag{37}$$

We multiply by $q_{\frac{p}{2}-1}$ to obtain

$$na_{\frac{p}{2}}q_{\frac{p}{2}-1}^3 + 2nq_{\frac{p}{2}-2}q_{\frac{p}{2}-1}^2 - a_{\frac{p}{2}}k_{\frac{p}{2}-1}^2q_{\frac{p}{2}-1} = 2k_{\frac{p}{2}-1}k_{\frac{p}{2}-2}q_{\frac{p}{2}-1} = 2k_{\frac{p}{2}-1}\left(k_{\frac{p}{2}-1}q_{\frac{p}{2}-2} + (-1)^{\frac{p}{2}-1}\right),$$

leading to

$$na_{\frac{p}{2}}q_{\frac{p}{2}-1}^3 + 2nq_{\frac{p}{2}-2}q_{\frac{p}{2}-1}^2 - a_{\frac{p}{2}}k_{\frac{p}{2}-1}^2q_{\frac{p}{2}-1} - 2k_{\frac{p}{2}-1}^2q_{\frac{p}{2}-2} = (-1)^{\frac{p}{2}-1}2k_{\frac{p}{2}-1}, \tag{38}$$

and

$$\left(nq_{\frac{p}{2}-1}^2 - k_{\frac{p}{2}-1}^2\right)\left(a_{\frac{p}{2}}q_{\frac{p}{2}-1} + 2q_{\frac{p}{2}-2}\right) = (-1)^{\frac{p}{2}-1}2k_{\frac{p}{2}-1}. \tag{39}$$

Therefore, it follows that

$$\frac{nq_{\frac{p}{2}-1}^2 - k_{\frac{p}{2}-1}^2}{2k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}} = \frac{(-1)^{\frac{p}{2}-1}}{a_{\frac{p}{2}}q_{\frac{p}{2}-1}^2 + 2q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}}. \tag{40}$$

If we use again the difference property (24), we get

$$\frac{nq_{\frac{p}{2}-1}^2 - k_{\frac{p}{2}-1}^2}{2k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}} = \frac{k_{\frac{p}{2}-2}q_{\frac{p}{2}-1} - k_{\frac{p}{2}-1}q_{\frac{p}{2}-2}}{a_{\frac{p}{2}}q_{\frac{p}{2}-1}^2 + 2q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}} = \frac{k_{\frac{p}{2}-1}q_{\frac{p}{2}-2} + k_{\frac{p}{2}-2}q_{\frac{p}{2}-1} - 2k_{\frac{p}{2}-1}q_{\frac{p}{2}-2}}{a_{\frac{p}{2}}q_{\frac{p}{2}-1}^2 + 2q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}}$$

$$= \frac{a_{\frac{p}{2}}k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}^2 + k_{\frac{p}{2}-1}q_{\frac{p}{2}-2}q_{\frac{p}{2}-1} + k_{\frac{p}{2}-2}q_{\frac{p}{2}-1}^2 - a_{\frac{p}{2}}k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}^2 - 2k_{\frac{p}{2}-1}q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}\left(a_{\frac{p}{2}}q_{\frac{p}{2}-1}^2 + 2q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}\right)}$$

$$= \frac{q_{\frac{p}{2}-1}\left(a_{\frac{p}{2}}k_{\frac{p}{2}-1}q_{\frac{p}{2}-1} + k_{\frac{p}{2}-1}q_{\frac{p}{2}-2} + k_{\frac{p}{2}-2}q_{\frac{p}{2}-1}\right) - k_{\frac{p}{2}-1}\left(a_{\frac{p}{2}}q_{\frac{p}{2}-1}^2 + 2q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}\right)}{q_{\frac{p}{2}-1}\left(a_{\frac{p}{2}}q_{\frac{p}{2}-1}^2 + 2q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}\right)}$$

$$= \frac{a_{\frac{p}{2}}k_{\frac{p}{2}-1}q_{\frac{p}{2}-1} + k_{\frac{p}{2}-1}q_{\frac{p}{2}-2} + k_{\frac{p}{2}-2}q_{\frac{p}{2}-1}}{a_{\frac{p}{2}}q_{\frac{p}{2}-1}^2 + 2q_{\frac{p}{2}-2}q_{\frac{p}{2}-1}} - \frac{k_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}}$$

$$= \frac{\left(a_{\frac{p}{2}} + \frac{q_{\frac{p}{2}-2}}{q_{\frac{p}{2}-1}}\right)k_{\frac{p}{2}-1} + k_{\frac{p}{2}-2}}{\left(a_{\frac{p}{2}} + \frac{q_{\frac{p}{2}-2}}{q_{\frac{p}{2}-1}}\right)q_{\frac{p}{2}-1} + q_{\frac{p}{2}-2}} - \frac{k_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}} = \left[a_0; a_1, \dots, a_{\frac{p}{2}-1}, a_{\frac{p}{2}} + \frac{q_{\frac{p}{2}-2}}{q_{\frac{p}{2}-1}}\right] - \frac{k_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}}$$

$$= \left[a_0; a_1, \dots, a_{\frac{p}{2}-1}, a_{\frac{p}{2}}, \frac{q_{\frac{p}{2}-1}}{q_{\frac{p}{2}-2}}\right] - \frac{k_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}} = \left[a_0; a_1, \dots, a_{\frac{p}{2}-1}, a_{\frac{p}{2}}, a_{\frac{p}{2}-1}, \dots, a_1\right] - \frac{k_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}}$$

$$= [a_0; a_1, \dots, a_{p-1}] - \frac{k_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}} = \frac{k_{p-1}}{q_{p-1}} - \frac{k_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}}.$$

We conclude that

$$\frac{k_{p-1}}{q_{p-1}} = \frac{nq_{\frac{p}{2}-1}^2 - k_{\frac{p}{2}-1}^2}{2k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}} + \frac{k_{\frac{p}{2}-1}}{q_{\frac{p}{2}-1}} = \frac{nq_{\frac{p}{2}-1}^2 + k_{\frac{p}{2}-1}^2}{2k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}},$$

when the length p is even and the statement follows. ■

For example, let us consider Pell's equation $4729494y^2 - x^2 = -1$ that appears when solving Archimedes's cattle problem. The continued fraction of $\sqrt{4729494}$ has period with length $p = 92$. By Theorem 5.1, the first non-trivial integer solution of this equation is (k_{91}, q_{91}), which would demand to calculate 91 convergents $\frac{k_i}{q_i}$. However, by Theorem 5.2, it follows that

$$\frac{4729494q_{45}^2 + k_{45}^2}{2k_{45}q_{45}} = \frac{k_{91}}{q_{91}}, \quad (41)$$

so, it is only required to calculate the first 45 convergents when using the method of continued fractions to obtain the solution.

Let us take another look to Theorem 5.2: it states that the point $f\left(\frac{k_*}{q_*}\right) = \left(\frac{nq_*^2+k_*^2}{nq_*^2-k_*^2}, \frac{2k_*q_*}{nq_*^2-k_*^2}\right)$ is proportional to the smallest non-trivial integer solution (k_s, q_s) of Pell's equation (3). By Remark 2.2, it follows that

$$\left(\frac{nq_*^2 + k_*^2}{nq_*^2 - k_*^2}, \frac{2k_*q_*}{nq_*^2 - k_*^2}\right) = (\pm k_s, \pm q_s). \tag{42}$$

Therefore, we can express the smallest non-trivial integer solution (k_s, q_s) of Pell's equation (3) fully in terms of the lower-order convergent $\frac{k_*}{q_*}$ as follows:

Proposition 5.3. *Let n be a natural number that is not a square and let us consider the continued fraction of $\sqrt{n}$ with period of length p and convergents $\frac{k_i}{q_i}$.*

(1) If p is odd, then the smallest non-trivial integer solution (k_s, q_s) of Pell's equation $ny^2 - x^2 = -1$ is

$$(k_{2p-1}, q_{2p-1}) = (nq_{p-1}^2 + k_{p-1}^2, 2k_{p-1}q_{p-1}). \tag{43}$$

(2) If p is even, then the smallest non-trivial integer solution (k_s, q_s) of Pell's equation $ny^2 - x^2 = -1$ is

$$(k_{p-1}, q_{p-1}) = \left(\frac{nq_{\frac{p}{2}-1}^2 + k_{\frac{p}{2}-1}^2}{nq_{\frac{p}{2}-1}^2 - k_{\frac{p}{2}-1}^2}, \frac{2k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}}{nq_{\frac{p}{2}-1}^2 - k_{\frac{p}{2}-1}^2}\right), \tag{44}$$

if $\frac{p}{2}$ is odd, and is

$$(k_{p-1}, q_{p-1}) = \left(\frac{k_{\frac{p}{2}-1}^2 + nq_{\frac{p}{2}-1}^2}{k_{\frac{p}{2}-1}^2 - nq_{\frac{p}{2}-1}^2}, \frac{2k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}}{k_{\frac{p}{2}-1}^2 - nq_{\frac{p}{2}-1}^2}\right), \tag{45}$$

if $\frac{p}{2}$ is even.

Proof. If the length p is odd, Theorem 5.1 guarantees that

$$nq_{p-1}^2 - k_{p-1}^2 = (-1)^{p-1} = 1, \tag{46}$$

so that the rational point

$$f\left(\frac{k_{p-1}}{q_{p-1}}\right) = \left(\frac{nq_{p-1}^2 + k_{p-1}^2}{nq_{p-1}^2 - k_{p-1}^2}, \frac{2k_{p-1}q_{p-1}}{nq_{p-1}^2 - k_{p-1}^2}\right) = (nq_{p-1}^2 + k_{p-1}^2, 2k_{p-1}q_{p-1}) \tag{47}$$

has positive entries and then Equation (43) follows.

On the other hand, if p is even, it follows from Theorem 5.2 and Remark 2.2 that

$$\left(\frac{nq_{\frac{p}{2}-1}^2 + k_{\frac{p}{2}-1}^2}{nq_{\frac{p}{2}-1}^2 - k_{\frac{p}{2}-1}^2}, \frac{2k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}}{nq_{\frac{p}{2}-1}^2 - k_{\frac{p}{2}-1}^2}\right) = (\pm k_{p-1}, \pm q_{p-1}), \tag{48}$$

where the signal depends on $\frac{p}{2}$. If $\frac{p}{2}$ is odd, then $\frac{p}{2}-1$ is even and then Lemma 4.1 guarantees that the point $\left(\frac{nq_{\frac{p}{2}-1}^2+k_{\frac{p}{2}-1}^2}{nq_{\frac{p}{2}-1}^2-k_{\frac{p}{2}-1}^2},\frac{2k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}}{nq_{\frac{p}{2}-1}^2-k_{\frac{p}{2}-1}^2}\right)$ lies in the first quadrant. Therefore, Equation (44) follows. If $\frac{p}{2}$ is even, then $\frac{p}{2}-1$ is odd and then the point $\left(\frac{nq_{\frac{p}{2}-1}^2+k_{\frac{p}{2}-1}^2}{nq_{\frac{p}{2}-1}^2-k_{\frac{p}{2}-1}^2},\frac{2k_{\frac{p}{2}-1}q_{\frac{p}{2}-1}}{nq_{\frac{p}{2}-1}^2-k_{\frac{p}{2}-1}^2}\right)$ lies in the third quadrant; hence, Equation (45) follows. ■

References

1. Chu W. Regular Polygonal Numbers and Generalized Pell Equations. *Int. Math. Forum.* 2007; 2(16):781-802. test.dvi (https://www.m-hikari.com/imf-password2007/13-16-2007/chuIMF13-16-2007.pdf)

2. Olds CD. Continued Fractions. Washington: Mathematical Association of America. 1963; 112-113, 19-28, 113-115 pp. Continued Fractions.pdf (https://www.ms.uky.edu/~sohum/ma330/files/Continued%20Fractions.pdf)

3. Rippon PJ, Taylor H. Even and Odd Periods in Continued Fractions of Square Roots. *The Fibonacci Quarterly*. 2004; 42(2):170-180. quartrippon02_2004.pdf (https://www.mathstat.dal.ca/FQ/Papers1/42-2/quartrippon02_2004.pdf)

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