Facultad de Ciencias Naturales y Exactas
Universidad del Valle

A Priori Estimates of the Prescribed Scalar Curvature on the Sphere

Gonzalo García Camacho
Universidad del Valle

Liliana Posada
Universidad del Valle

Recived: February 9, 2015 Accepted: July 23, 2015

Pag. 73-86

Abstract

This paper considers the prescribed scalar curvature problem on the sphere for $n \geq 3$. Given a prescribed scalar curvature function $K : S^n \to \mathbb{R}$ and a centered dilation defined by $F_y = \Sigma^{-1} \circ D_\beta \circ \Sigma$, $y \in B^{n+1}$, where Σ is the stereographic projection and D_β is a dilation in $\mathbb{R}^n$, in this work we estimate the gradient of the function K near the critical point of the function $\overline{J}_p(y) = \int_{S^n} K(\zeta)\phi^{p+1} d\sigma(\zeta)$ where $\phi(y) = |(F_y^{-1})'|^{\frac{n-2}{2}}$. We will use this estimate to find L^p estimates of the first two y-derivatives of the function $K \circ F_y(\xi)$.

Keywords: metrics, scalar curvature, conformal geometry.

1 Introduction

Let (S^n, δ_{ij}) be the unitary sphere with the standard metric. A natural question in Riemannian geometry is: Given a function $K : S^n \to \mathbb{R}$, is there a metric g conformally related to the standard metric δ_{ij} such that K is the scalar curvature of S^n with respect to the metric g? This is equivalent to the problem of finding a positive smooth function $u : S^n \to \mathbb{R}$ which satisfies the equation:

$$\Delta u - \frac{n(n-2)}{4} u + \frac{n-2}{4(n-1)} K u^{\frac{n+2}{n-2}} = 0. \tag{1}$$

If we set $g = u^{\frac{4}{n-2}} \delta_{ij}$, where u is a solution of this problem, then the function K is the scalar curvature of S^n with respect to the metric g.

The problem of conformal deformation of metrics in S^n have been extensively studied by many authors (for example, see [1, 2, 3, 4, 5, 6, 7, 8] and the references therein). An important feature of this problem is that it is a conformal invariant one. More precisely, if u is a solution of equation (1), then for any conformal map $F : S^n \to S^n$ the function $\alpha_F(u) = |(F^{-1})'|^{\frac{n-2}{2}} u \circ F^{-1}$ is a solution to problem (1) with scalar curvature $K \circ F$.

The problem of conformal deformation of metrics in S^n can be approached using the so called Yamabe method, which consists in studying first the subcritical problem in the equation (1):

$$\Delta u_p - \frac{n(n-2)}{4} u_p + \frac{n-2}{4(n-1)} K u_p^p = 0, \tag{2}$$

with $p \in \left(1, \frac{n+2}{n-2}\right)$ and then consider the limit of the solutions when $p \uparrow \frac{n+2}{n-2}$.

Let $E(u)$ be the energy norm associated with the linear part of (2), that is:

$$E(u) = \int_{S^n} \left(\frac{n(n-2)}{4} u^2 + |\nabla u|^2 \right) d\sigma_g,$$

and let

$$\mathcal{S} = \{u \in H^1(S^n) \text{ with } u \geq 0 \text{ almost everywhere and with } E(u) = E(1)\}.$$

Let us consider the open unit ball B^{n+1} and the map $\Phi : B^{n+1} \to \mathcal{S}$ defined by:

$$\Phi(y) = \alpha_y := \alpha_{F_y}(1) = |(F_y^{-1})'|^{\frac{n-2}{2}},$$

where $F_y : S^n \to S^n$ is the restriction to S^n of a special conformal map $F_y : \overline{B^{n+1}} \to \overline{B^{n+1}}$ that satisfies $F_y(0) = y$ and fix the points $\pm \frac{y}{|y|}$; this function maps 0 to y and commutes with rotations about the line joining the origin and the point y. This map is referred to as a centered dilation.

For $p \in \left(1, \frac{n+2}{n-2}\right)$ and $u \in \mathcal{S}$, let $J_p(u)$ defined by:

$$J_p(u) = \int_{S^n} K u^{p+1} d\sigma.$$

If u is a critical point of $J_p(.)$ on $\mathcal{S}$, then a multiple of u satisfies problem (2). Let us define the function $\overline{J}_p = J_p \circ \Phi$.

This work is motivated by the work of Schoen and Zhang [7] on the prescribed scalar curvature problem on the n-dimensional sphere, $n \geq 3$, where they prove an existence result for $n = 3$.

To determine the nature of the critical points of J_p, they study the critical points of $\overline{J}_p$ and then make a perturbation argument. In order to understand the nature of the critical points of $\overline{J}_p$ near the border of B^{n+1}, they study the behavior of K near those critical points.

In this paper we will study more closely the behavior of the function K near the critical points of $\overline{J}_p$. Given a critical point y_0 of $\overline{J}_p$, near S^n, in this work we will find an estimate of the gradient of the function K at the point $\frac{y_0}{|y_0|}$ and we will use this to find L^p estimates of the function $f \circ F_y(\xi) = \left(K \circ F_y(\xi) - K\left(\frac{y}{|y|}\right)\right)$ and its first two y-derivatives. Our method to get the estimates parallels that of Escobar and Garcia ([3]) in the problem of prescribed mean curvature on the boundary of the ball. In a coming paper we will use the estimates found in this work to give an alternative proof for some of the results in [7].

2 Preliminaries

Let $y \in B^{n+1}$. Up to a rotation we will assume that $y = (0, ..., 0, y_{n+1})$. In this case the centered dilation function F_y is given by $F_y(x) = \Sigma^{-1} \circ D_\beta \circ \Sigma(x)$, where the functions Σ and D_β are defined as follows. The function $\Sigma : \overline{B^{n+1}} \setminus \{(0,0,\ldots,0,-1)\} \to \overline{\mathbb{R}^{n+1}_+}$ is defined as $\Sigma = T_{-2} \circ I_2 \circ T_1$ where $T_a(x_1,\ldots,x_{n+1}) = (x_1,\ldots,x_n,x_{n+1}+a)$, and I_R is the inversion map $x \to \frac{R^2 x}{|x|^2}$. Here $\mathbb{R}^{n+1}_+$ denotes the upper half $(n+1)$-dimensional Euclidean space.

Hence,

$$\Sigma(x) = \left(\frac{4\overline{x}}{|\overline{x}|^2 + (1+x_{n+1})^2}, \frac{2(1-|\overline{x}|^2 - x_{n+1}^2)}{|\overline{x}|^2 + (1+x_{n+1})^2}\right),$$

where $\overline{x} = (x_1,\ldots,x_n)$ and $x = (\overline{x}, x_{n+1})$. Observe that if $|x| = \sqrt{|\overline{x}|^2 + x_{n+1}^2} = 1$, then

$$\Sigma(x) = \left(\frac{2\overline{x}}{1+x_{n+1}}, 0\right).$$

Thus the map $\Sigma|_{S^n - \{(0,0,\ldots,0,-1)\}}$ is the stereographic projection. The inverse function of Σ is:

$$\Sigma^{-1}(x) = \left(\frac{4\overline{x}}{|\overline{x}|^2 + (2+x_{n+1})^2}, \frac{4-|x|^2}{|\overline{x}|^2 + (2+x_{n+1})^2}\right).$$

When $x_{n+1} = 0$ we get

$$\Sigma^{-1}(\overline{x},0) = \left(\frac{4\overline{x}}{|\overline{x}|^2+4}, \frac{4-|\overline{x}|^2}{|\overline{x}|^2+4}\right),$$

which is the inverse of the stereographic projection from the south pole of the sphere.

The function $D_\beta : \overline{\mathbb{R}^{n+1}_+} \to \overline{\mathbb{R}^{n+1}_+}$ is defined by $D_\beta(x) = \beta x$, where $\beta = \frac{1-|y|}{1+|y|}$.

Proposition 2.1. *If $F_y = \Sigma^{-1} \circ D_\beta \circ \Sigma$ then*

$$F_y(x) = B^{-1}(4\beta A\overline{x}, (A^2 - 4\beta^2|\overline{x}|^2 - \beta^2(1-|x|^2)))$$

and $F_y(0) = y$, where $\beta = \frac{1-|y|}{1+|y|}$, $x = (x_1,\ldots,x_{n+1})$, $\bar{x} = (x_1,\ldots,x_n)$,

$$A = |\overline{x}|^2 + (1+x_{n+1})^2 \quad \text{and} \quad B = 4\beta^2|\overline{x}|^2 + [A + \beta(1-|x|^2)]^2.$$

Proof. Since

$$D_\beta \circ \Sigma(x) = \left(\frac{4\beta\overline{x}}{|\overline{x}|^2 + (1+x_{n+1})^2}, \frac{2\beta(1 - x_{n+1}^2 - |\overline{x}|^2)}{|\overline{x}|^2 + (1+x_{n+1})^2} \right),$$

then

$$\begin{aligned} F_y(x) &= \Sigma^{-1}\left(\frac{4\beta\overline{x}}{|\overline{x}|^2 + (1+x_{n+1})^2}, \frac{2\beta(1 - x_{n+1}^2 - |\overline{x}|^2)}{|\overline{x}|^2 + (1+x_{n+1})^2} \right) \\ &= \Sigma^{-1}(4\beta A^{-1}\overline{x}, 2\beta A^{-1}(1 - x_{n+1}^2 - |\overline{x}|^2)) \\ &= B^{-1}(4\beta A\overline{x}, (A^2 - 4\beta^2|\overline{x}|^2 - \beta^2(1-|x|^2))). \end{aligned}$$

Therefore

$$F_y(x) = B^{-1}(4\beta A\overline{x}, (A^2 - 4\beta^2|\overline{x}|^2 - \beta^2(1-|x|^2))).$$

If $x = 0$, then $A = 1$ and $B = (1+\beta)^2$. Hence,

$$F_y(0) = [(1+\beta)^{-2}](0, 1-\beta^2) = \left(0, \frac{1-\beta^2}{(1+\beta)^2}\right) = \left(0, \frac{1-\beta}{(1+\beta)}\right) = (0, y_{n+1}) = y.$$

Since $F_y(0) = y$ then $\beta = \frac{1-|y|}{1+|y|}$ when $y_{n+1} \geq 0$ and $\beta = \frac{1+|y|}{1-|y|}$ when $y_{n+1} \leq 0$. For this paper we will use the convention that when β is large, we call it λ and when β is small we call it μ. Observe that $F_y^{-1} = F_{-y}$. In order to get the L^p estimates of the derivatives of the function $K \circ F_y$, we need the estimates of the derivatives of the function F_y. If we rewrite the function F_y as

$$F_y(z) = \frac{(4\mu\bar{z}|z-s|^2, \;\; |z-s|^4 - 4\mu^2|z'|^2)}{|z-s|^4 + 4\mu^2|\bar{z}|^2},$$

where $\bar{z} = (z_1, z_2, \ldots, z_n)$, $z = (\bar{z}, z_{n+1})$, the calculations in [3] leads to

Lemma 2.2. *For* $1 \leq i, j \leq n+1$

$$\left| \frac{\partial F_y}{\partial y_i}(z) \right| \leq \frac{C}{\mu^r |z-s|^{1-r}}, \tag{3}$$

and

$$\left| \frac{\partial^2 F_y}{\partial y_j \partial y_i} \right| \leq \frac{C_1}{\mu^r |z-s|^{2-r}} + \frac{C_2}{\mu^r |z-s|^{1-r}},$$

where $z \in S^n$, $s = (\overline{0}, -1)$, $\mu = \frac{1-|y|}{1+|y|}$ *and* $0 \leq r \leq 1$.

Proposition 2.3. *For* $x \in S^n$, *we get*

$$F_y^*(\delta_{ij})_x = \left(\frac{1-|y|^2}{|y+x|^2} \right)^2 \delta_{ij},$$

where $F_y^*(\delta_{ij})$ *is the pullback of the metric* δ_{ij} *induced by the function* F_y.

Proof. Given $x \in S^n$, straightforward calculations show that

$\Sigma^*(\delta_{ij})_x(e_i, e_j) =< \frac{\partial \Sigma}{\partial x_i}, \frac{\partial \Sigma}{\partial x_j} >= \frac{4}{(1+x_{n+1})^2}\delta_{ij}$, $D^*_\beta(\delta_{ij})_{\Sigma(x)}(e_i, e_j) = \beta^2\delta_{ij}$,
and

$$(\Sigma^{-1})^*(\delta_{ij})_{\beta\Sigma(x)}(e_i, e_j) = \frac{16}{\left(\left|\frac{4\beta\overline{x}}{|\overline{x}|^2 + (1+x_{n+1})^2}\right|^2 + 4\right)^2}\delta_{ij}.$$

Hence,

$$(\Sigma^{-1})^*(\delta_{ij})_{\beta\Sigma(x)}(e_i, e_j) = \frac{(1+x_{n+1})^2}{(\beta^2(1-x_{n+1}) + (1+x_{n+1}))^2}\delta_{ij}.$$

Since the metrics $\Sigma^*(\delta_{ij})$, $(\Sigma^{-1})^*$ and D^*_β are diagonal, then

$$F_y{}^*(\delta_{ij}) = \Sigma^*(\delta_{ij}).D_\beta{}^*(\delta_{ij}).\Sigma^{-1*}(\delta_{ij}).$$

Thus,

$$F^*_y(\delta_{ij})(e_i, e_j) = \frac{(1+x_{n+1})^2}{(\beta^2(1-x_{n+1}) + (1+x_{n+1}))^2}.\frac{4}{(1+x_{n+1})^2}.\beta^2\delta_{ij}$$

hence,

$$F^*_y(\delta_{ij})(e_i, e_j) = \frac{4\beta^2}{(\beta^2(1-x_{n+1}) + (1+x_{n+1}))^2}\delta_{ij}.$$

If $y_{n+1} \geq 0$, then $\beta = \frac{1-|y|}{1+|y|}$ where $|y| = y_{n+1}$ and consequently

$$\begin{aligned}\beta^2(1-x_{n+1}) + (1+x_{n+1}) &= \left(\frac{1-|y|}{1+|y|}\right)^2 (1-x_{n+1}) + (1+x_{n+1}) \\ &= \frac{(1-|y|)^2(1-x_{n+1}) + (1+|y|)^2(1+x_{n+1})}{(1+|y|)^2} \\ &= \frac{2(1+|y|^2+2|y|x_{n+1})}{(1+|y|)^2} = \frac{2|y+x|^2}{(1+|y|)^2}.\end{aligned}$$

Then,

$$\begin{aligned}F^*_y(\delta_{ij})(e_i, e_j) &= \frac{4\beta^2}{(\beta^2(1-x_{n+1}) + (1+x_{n+1}))^2}\delta_{ij} = \frac{((1+|y|)^2)^2\frac{(1-|y|)^2}{(1+|y|)^2}}{(|y+x|^2)^2}\delta_{ij} \\ &= \frac{(1-|y|^2)^2}{(|y+x|^2)^2}\delta_{ij} = \left(\frac{1-|y|^2}{|y+x|^2}\right)^2\delta_{ij}.\end{aligned}$$

From now on, we will denote for $|(F_y)'|(\zeta) = \dfrac{1-|y|^2}{|y+\zeta|^2}$, $\zeta \in S^n$, the linear stretch factor of the conformal transformation F_y.

Proposition 2.4. *In stereographic coordinates*

$$|y-\zeta|^2 = \frac{4(|\overline{x}|^2+4\mu^2)}{(|\overline{x}|^2+4)(1+\mu^2)},$$

where $\mu = \frac{1-|y|}{1+|y|}$, *and* $\zeta \in S^n$.

Proof.

$$\begin{aligned}
|y-\zeta|^2 = |y|^2 - 2y.\zeta + |\zeta|^2 = 1 - 2|y|.\left(\frac{4-|\overline{x}|^2}{|\overline{x}|^2+4}\right) + |y|^2 \\
= \frac{(1+|y|^2)(|\overline{x}|^2+4) - 2|y|(4-|\overline{x}|^2)}{|\overline{x}|^2+4} \\
= \frac{(1+|y|)^2|\overline{x}|^2 + 4(1-|y|)^2}{|\overline{x}|^2+4},
\end{aligned}$$

and therefore,

$$|y-\zeta|^2 = \frac{4(|\overline{x}|^2+4\mu^2)}{(|\overline{x}|^2+4)(1+\mu^2)},$$

where in the last equality, we have used $|y| = \frac{1-\mu}{1+\mu}$.

2 Main Result

The main purpose of this work is to give the following estimate for the gradient of the prescribed scalar curvature function K near a critical point of the function $\overline{J}_p$.

Theorem 3.1. *Let* y *be a critical point of the function* $\overline{J}_p$ *near* S^n, *then,*

$$\text{if } n=3, \quad \left|\nabla K\left(\frac{y}{|y|}\right)\right| \le C\mu^{1-w}$$

and

$$\text{if } n \ge 4, \quad \left|\nabla K\left(\frac{y}{|y|}\right)\right| \le C\mu^{2-w},$$

where w *is any small positive number less than one.*

Proof. Let us take rectangular coordinates in $\mathbb{R}^{n+1}$ such that $y = (0,0,\ldots,|y|)$, then $|y|^{-1}y = (0,0,\ldots,|y|^{-1}|y|) = (0,0,\ldots,1) = N$. Since

$$\overline{J}_p(y) = \int_{S^n} K(\zeta)\left(\frac{1-|y|^2}{|y-\zeta|^2}\right)^{\frac{n-2}{2}(p+1)} d\sigma(\zeta),$$

then,

$$\frac{\partial \overline{J}_p}{\partial y_j} = (2-n)(p+1)\int_{S^n} K(\zeta)\left(\frac{1-|y|^2}{|y-\zeta|^2}\right)^{\frac{n-2}{2}(p+1)} \left[\frac{y_j}{1-|y|^2} + \frac{(y_j-\zeta_j)}{|y-\zeta|^2}\right] d\sigma(\zeta).$$

Evaluating at the critical point y we find that for $j = 1, \dots, n$

$$0 = \frac{\partial \overline{J}_p}{\partial y_j}(y) = (2-n)(p+1)\int_{S^n} K(\zeta)\left(\frac{1-|y|^2}{|y-\zeta|^2}\right)^{\frac{n-2}{2}(p+1)}\left(\frac{-\zeta_j}{|y-\zeta|^2}\right) d\sigma(\zeta)$$

and therefore,

$$\int_{S^n} K(\zeta)\left(\frac{1-|y|^2}{|y-\zeta|^2}\right)^{\frac{n-2}{2}(p+1)}\left(\frac{\zeta_j}{|y-\zeta|^2}\right) d\sigma(\zeta) = 0.$$

The last integral also vanishes when K is a constant. Thus,

$$\int_{S^n} K(N)\left(\frac{1-|y|^2}{|y-\zeta|^2}\right)^{\frac{n-2}{2}(p+1)}\left(\frac{\zeta_j}{|y-\zeta|^2}\right) d\sigma(\zeta) = 0.$$

Hence,

$$\int_{S^n} (K(\zeta)-K(N))\left(\frac{1-|y|^2}{|y-\zeta|^2}\right)^{\frac{n-2}{2}(p+1)}\left(\frac{\zeta_j}{|y-\zeta|^2}\right) d\sigma(\zeta) = 0.$$

This equality, in the stereographic coordinates is equivalent to:

$$\int_{\mathbb{R}^n} \frac{(K(x)-K(0))x_j}{4^{n+1-\frac{n-2}{2}\delta}}\left(\frac{(1+\mu)^2(|x|^2+4)}{|x|^2+4\mu^2}\right)^{n+1-\frac{n-2}{2}\delta}\frac{2^n}{(4+|x|^2)^n}dx = 0,$$

where $\delta = \frac{n+2}{n-2} - p$ is a small positive number.

The transformation $y = \mu x$ yields

$$\int_{\mathbb{R}^n} \frac{(K(\mu x)-K(0))x_j dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta-1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}} = 0. \tag{4}$$

By Taylor's Theorem, there exists $\delta_0 > 0$ small enough such that for $|x| \le \mu^{-1}\delta_0$ we have:

$$K(\mu x) - K(0) = \sum_i \frac{\partial K}{\partial x_i}(0)\mu x_i + \frac{1}{2}\sum_{i,k}\frac{\partial^2 K}{\partial x_i \partial x_k}(0)\mu^2 x_i x_k + O(\mu^3|x|^3).$$

It is easy to check that

$$\left|\int_{B_{\mu^{-1}\delta_0}(0)} \frac{x_j^2 dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta-1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}}\right| \ge C,$$

and therefore

$$\left| \int_{B_{\mu^{-1}\delta_0}(0)} \mu \frac{\sum_i \frac{\partial K}{\partial x_i}(0) x_i x_j dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta-1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}} \right| \geq C\mu \left| \frac{\partial K}{\partial x_j}(0) \right|.$$

On the other hand,

$$\frac{1}{2}\mu^2 \int_{B_{\mu^{-1}\delta_0}(0)} \frac{\sum_{i,k} \frac{\partial^2 K}{\partial x_i \partial x_k}(0) x_i x_k x_j dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta-1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}} = 0$$

because of the symmetries of the ball, and a straightforward calculation yields:

$$O\left(\mu^3 \int_{B_{\mu^{-1}\delta_0}(0)} \frac{|x|^3 x_j dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta-1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}} \right) \leq C\mu^{3-w},$$

where w is a small positive number, and

$$\left| \int_{\mathbb{R}^n \setminus B_{\mu^{-1}\delta_0}(0)} \frac{(K(\mu x)-K(0)) x_j dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta-1}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}} \right|$$

$$\leq C \int_{\mathbb{R}^n \setminus B_{\mu^{-1}\delta_0}(0)} \frac{|x|(4+\mu^2|x|^2) dx}{(\mu^2|x|^2+4)^{\frac{n-2}{2}\delta}(|x|^2+4)^{n+1-\frac{n-2}{2}\delta}} \leq C\mu^{n-1-(n-2)\delta}.$$

The last inequalities and equality (4) imply:

$$C\mu \left| \frac{\partial K}{\partial x_j}(0) \right| \leq C\mu^{3-w} + C\mu^{n-1-(n-2)\delta}.$$

Then if $n = 3$

$$\left| \frac{\partial K}{\partial x_j}(0) \right| \leq C\mu^{2-w} + C\mu^{1-\delta} \leq C\mu^{1-w},$$

and if $n \geq 4$,

$$\left| \frac{\partial K}{\partial x_j}(0) \right| \leq C\mu^{2-w}.$$

In the following propositions, we will use this estimate to find some estimates on the function $K(F_y((\xi)) - K\left(\frac{y}{|y|}\right)$ and the first y- derivatives of the function $K \circ F_y$.

Lemma 3.2. *Let y be a critical point of $\overline{J}_p$ near S^n and let $f = K - K\left(\frac{y}{|y|}\right)$. If $1 \leq q < n$, then, $\|f \circ F_y\|_{0,q} \leq C\mu^{2-w}$ for some $0 < w < 1$.*

Proof. Taylor's Theorem yields:

$$\begin{aligned} |f\circ F_y(\xi)| &= \left|K(F_y(\xi)) - K\left(\frac{y}{|y|}\right)\right| \\ &\leq C_1\left|\nabla K\left(\frac{y}{|y|}\right)\right|\left|F_y(\xi)-\frac{y}{|y|}\right| + C_2\left|F_y(\xi)-\frac{y}{|y|}\right|^2, \end{aligned}$$

in a geodesic ball of radius r and center $\frac{y}{|y|}$. Here C_1 and C_2 denote positive constants.

Then,

$$|f\circ F_y(\xi)|^q \leq C_1\left|\nabla K\left(\frac{y}{|y|}\right)\right|^q\left|F_y(\xi)-\frac{y}{|y|}\right|^q + C_2\left|F_y(\xi)-\frac{y}{|y|}\right|^{2q}.$$

Since

$$\|f\circ F_y\|_{0,q}^q = \int_{S^n}|f\circ F_y(\xi)|^q d\sigma_g = \int_V |f\circ F_y(\xi)|^q d\sigma_g + \int_{S^n\setminus V}|f\circ F_y(\xi)|^q d\sigma_g,$$

then it follows that

$$\begin{aligned} \left(\int_{S^n}|f\circ F_y(\xi)|^q d\sigma_g\right)^{1/q} &= \left(\int_V |f\circ F_y(\xi)|^q d\sigma_g + \int_{S^n\setminus V}|f\circ F_y(\xi)|^q d\sigma_g\right)^{1/q} \\ &\leq 2^q\left(\int_V |f\circ F_y(\xi)|^q d\sigma_g\right)^{1/q} \\ &\quad + 2^q\left(\int_{S^n\setminus V}|f\circ F_y(\xi)|^q d\sigma_g\right)^{1/q}. \end{aligned}$$

On the one hand,

$$\begin{aligned} \left(\int_V |f\circ F_y(\xi)|^q d\sigma_g\right)^{1/q} &\leq C_1\mu^{1-w}\left(\int_V\left|F_y(\xi)-\frac{y}{|y|}\right|^q d\sigma_g\right)^{1/q} \\ &\quad + C_2\left(\int_V\left|F_y(\xi)-\frac{y}{|y|}\right|^{2q} d\sigma_g\right)^{1/q}. \end{aligned}$$

By Holder's inequality

$$\left(\int_V\left|F_y(\xi)-\frac{y}{|y|}\right|^q d\sigma_g\right)^{1/q} \leq C\left(\int_V\left|F_y(\xi)-\frac{y}{|y|}\right|^{2q}\right)^{1/2q}.$$

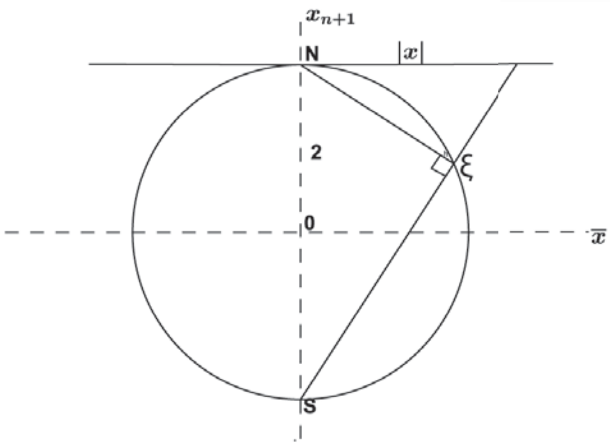
Then,

$$\left(\int_V |f\circ F_y(\xi)|^q\right)^{1/q} \leq M_1\mu^{1-w}\left[\left(\left(\int_V \left|F_y(\xi)-\frac{y}{|y|}\right|^{2q} d\sigma_g\right)^{1/q}\right]^{1/2} + M_2\left(\int_V \left|F_y(\xi)-\frac{y}{|y|}\right|^{2q} d\sigma_g\right)^{1/q}$$

Assume that $\frac{y}{|y|}$ is the north pole N. Since $F_y(N)=N$, in stereographic coordinates $|F_y(\xi)-N| = \dfrac{4^q|\mu x|^{2q}}{(4+|\mu x|^2)^q}$ (see figure below). Therefore, the second integral on the right hand is equivalent in stereographic coordinates to:

$$\int_{B_{R_1}} \frac{4^q|\mu x|^{2q}}{(4+|\mu x|^2)^q}\frac{2^n dx}{(4+|x|^2)^n},$$

where B_{R_1} is the image of the geodesic ball V under the stereographic projection.



Hence,

$$\int_V |F_y(\xi)-N|^{2q}\, d\sigma_g \leq C\int_{B_{R_1}} \mu^{2q}\left(\frac{|x|^2}{4+|x|^2}\right)^q dx \leq C\mu^{2q}.$$

Consequently,

$$\left(\int_V |f\circ F_y(\xi)|^q\right)^{1/q} \leq C\mu^{2-w}.$$

On the other hand

$$\int_{S^n\setminus V} |f\circ F_y(\xi)|^q d\sigma_g = \int_{F_y(S^n\setminus V)} |f(\zeta)|^q|(F_y^{-1})'|^n d\sigma_g(\zeta) \leq C\int_{\mathbb{R}^n\setminus B_{R_1}} \frac{\lambda^n}{(1+\lambda^2|x|^2)^n}dx,$$

where $\lambda = \frac{1+|y|}{1-|y|} = \mu^{-1}$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_{R_1}} \frac{\lambda^n}{(1+\lambda^2|x|^2)^n} dx &= \lambda^n \int_{R_1}^{\infty} \frac{r^{n-1}dr}{(1+\lambda^2 r^2)^n} \\ &\leq C\lambda^n \int_{R_1}^{\infty} \frac{r^{n-1}dr}{\lambda^{2n} r^{2n}} \\ &\leq C\lambda^{-n} \int_{R_1}^{\infty} r^{-n-1} dr.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^n \setminus B_{R_1}} \frac{\lambda^n}{(1+\lambda^2|x|^2)^n} dx \leq C\lambda^{-n} = C\mu^n,$$

and

$$\left(\int_{S^n \setminus V} |f \circ F_y(\xi)|^q d\sigma_g\right)^{1/q} \leq C\mu^{n/q} = C\mu^{2-\gamma}.$$

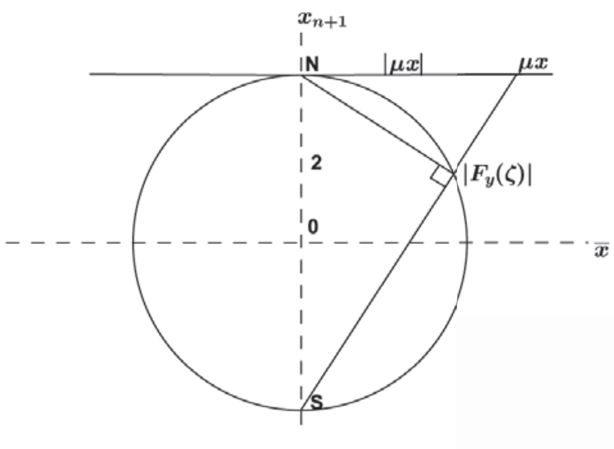
It follows that:

$$\|f \circ F_y(\xi)\|_{0,q} \leq C\mu^{2-w'}.$$

Proposition 3.3. *Let y_0 be a critical point of $\overline{J}_p$ near S^n, and let a small ball $B_{y_0} \subseteq B^{n+1}$, if $y \in B_{y_0}$ and $q \in \left(\frac{n}{2}, n\right)$ then, $\|\nabla_y(K \circ F_y)\|_{0,q} \leq C\mu^{1-w^*}$, where $\mu = \frac{1-|y_0|}{1+|y_0|}$ and w^* is a positive number less than one.*

Proof. Taking $r = 0$ in the last lemma, we get

$$\begin{aligned}I = \int_{S^n} |\nabla_y(K \circ F_y)|^q d\sigma &\leq \int_{S^n} |\nabla K(F_y)|^q |\nabla_y F_y|^q d\sigma \\ &\leq C \int_{S^n} \frac{|\nabla K(F_y)|^q}{|z-s|^q} d\sigma(z),\end{aligned}$$



Using in the above figure similarity of triangles, we find that $|z-s| = \dfrac{4}{\sqrt{4+|x|^2}}$. Consequently, the integral in the right is equivalent in stereographic coordinates to

$$\int_{R^n} \frac{(4+|x|^2)^{q/2}|\nabla K(\mu x)|^q}{4^q} \frac{2^n}{(4+|x|^2)^n} dx.$$

By Taylor's Theorem there exists $D > 0$ such that if $|x| \leq R = D\mu^{-1}$, then, we have

$$|\nabla K(\mu x)|^q \leq C|\nabla K(0)|^q + C|\mu x|^q.$$

Therefore,

$$I \leq C\left(\int_{B_R(0)} \frac{|\nabla K(0)|^q + |\mu x|^q}{(4+|x|^2)^{n-q/2}} dx + \int_{\mathbb{R}^{n-1}\setminus B_R(0)} \frac{dx}{(4+|x|^2)^{n-q/2}}\right).$$

On the one hand, taking α such that $n-\alpha$ is a very small positive number and using $q < n$ we get:

$$\begin{aligned}\int_{B_R(0)} \frac{|\nabla K(0)|^q}{(4+|x|^2)^{n-q/2}} dx &\leq C\mu^{q(1-w)} \int_{B_R(0)} \frac{dx}{(4+|x|^2)^{n-q/2}} \\ &\leq C\mu^{q(1-w)} \int_0^{D\mu^{-1}} r^{n-1-\alpha}\left(\frac{(r^2)^{\alpha/2}}{(4+r^2)^{n-q/2}}\right) dr \\ &\leq C\mu^{q-w_0}\end{aligned}$$

where $w_0 = -wq + \alpha - n$. It's easy check

$$\int_{B_R(0)} \frac{2^n(4+|x|^2)^{q/2}|\mu x|^q}{4^q(4+|x|^2)^n} dx \leq C\mu^q \int_0^{D\mu^{-1}} r^{-n-1+2q} dr \leq \frac{C}{2q-n}\mu^{n-q}.$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^{n-1}\setminus B_R(0)} \frac{dx}{(4+|x|^2)^{n-q/2}} &\leq \lim_{b\to\infty} \int_{C\mu^{-1}}^b \frac{dx}{(4+|x|^2)^{n-q/2}} \\ &\leq \lim_{b\to\infty} \int_{C\mu^{-1}}^b \frac{r^{n-1}dr}{r^{2(n-q/2)}} = \lim_{b\to\infty} \int_{C\mu^{-1}}^b r^{-1-n+q} dr \\ &\leq \frac{C}{n-q}\mu^{n-q}.\end{aligned}$$

Letting $w^* = \min\{\frac{w_0}{q}, 1 - \frac{n-q}{q}\}$, the estimate follows from the above inequalities.

Proposition 3.4. *If $q < n$ and $1 - \frac{n}{2q} < r < \frac{1}{2}$, then the following estimate holds:*

$$\|\nabla_y \nabla_y (K \circ F_y)\|_{0,q} \leq C\mu^{-2r}.$$

Proof. By the previous estimates of $\|\nabla_y (K \circ F_y)$ and $\|\nabla_y \nabla_y (K \circ F_y)\|$, we get

$$\begin{aligned} |\nabla_y \nabla_y (K \circ F_y)|^q &\leq C(|\nabla_y F_y|^2 + |\nabla_y \nabla_y F_y|)^q \\ &\leq \left[\left(\frac{C_1}{\mu^r |z-s|^{1-r}} \right)^2 + \left(\frac{C_2}{\mu^r |z-s|^{1-r}} \right) \right]^q. \end{aligned}$$

Using Holder's inequality, to get the desired estimate is enough to estimate the integral

$$\int_{S^n} \frac{d\sigma(z)}{\mu^{2rq} |z-s|^{(2-2r)q}} = \frac{C}{\mu^{2rq}} \int_{R^n} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx.$$

But

$$\int_{R^n} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx = \int_{B_1(0)} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx + \int_{R^n \setminus B_1(0)} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx.$$

Let us estimate the first integral in the right side

$$\int_{B_1(0)} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx = C \int_0^1 \frac{t^{n-1}}{(4+t^2)^{n-(1-r)q}} dt = C \int_0^1 \frac{t^{n-1}}{(4+t^2)^{n+(r-1)q}} dt.$$

Since $q < n$ and $1 - \frac{n}{2q} < r$ taking $\nu > 0$ such that $0 < n - \nu < 1$, then we have:

$$\int_0^1 \frac{t^{n-1-\nu}(t^2)^{\nu/2}}{(4+t^2)^{n+(r-1)q}} dt \leq C \int_0^1 t^{n-\nu-1} dt = \frac{C}{n-\nu},$$

where we have used that $\frac{\nu}{2} < \frac{n}{2} < n - (1-r)q$.
On the other hand,

$$\int_{R^n \setminus B_1(0)} \frac{(4+|x|^2)^{(1-r)q}}{(4+|x|^2)^n} dx = C \int_1^\infty \frac{t^{n-1}}{(4+t^2)^{n+(r-1)q}} dt = \frac{C}{2(r-1)q+n}$$

Consequently,

$$\left(\int_{S^n} |\nabla_y \nabla_y (K \circ F_y)|^q d\sigma(z) \right)^{1/q} \leq C\mu^{-2r}.$$

Acknowledgements. Gonzalo Garcia and Liliana Posada thanks to the Universidad del Valle for its support with the 7920 Project. The authors are very grateful to the reviewers for their suggestions and comments that really improved the paper.

References

[1] Bahri A. and Coron J. M. (1991). The scalar curvature problem on the standard three-dimensional sphere.J. Funct. Anal.95 106-172

[2] Chang A., Gursky M. and Yang P. (1993). The scalar curvature equation on the 2- and 3-sphere.Calc. Var.1 205-229.

[3] Escobar J. F. and García G. (2004). Conformal metrics on the ball with zero scalar curvature and prescribed mean curvature on the boundary. Journal of Functional Analysis 211 71-152.

[4] Han Z. C. (1991). Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent.Ann. Inst. Henri Poincaré anal. non linéaire 159-175.

[5] Kazdan J. and Warner F. (1975). Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature. Ann. Math.101 317-331.

[6] Li Y. Y. (1995). Prescribing scalar curvature on S^n and related problems. I. J. Differential Equations 120 , Nª 2, 319-410.

[7] Shoen R. and Zhang D. (1996). Prescribed scalar curvature on the n-sphere. Springer- Verlag .

[8] Zhang D. (1990). New results on geometric variational problems. Ph.D. Thesis, Stanford.

Author's address

Gonzalo García Camacho
Departamento de Matemáticas, Universidad del Valle, Cali - Colombia
gonzalo.garcia@correounivalle.edu.co

Liliana Posada Vera
Departamento de Matemáticas, Universidad del Valle, Cali - Colombia
liliana.posada@correounivalle.edu.co